NOTE ON MIXED VOLUMES OF CURRENTS

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Contents

1. Introduction

In this short note, we prove that the mixed volumes in the sense of Cao coincide with those defined by $\mathcal{I}$-envelopes.

2. Mixed volumes

Let X be a connected compact Kähler manifold of dimension n. Let $T_1, \dots, T_n$ be closed positive $(1,1)$-currents on X. Let $\theta_1, \dots, \theta_n$ be closed real smooth $(1,1)$-forms on X in the cohomology classes of $T_1, \dots, T_n$ respectively. Consider $\varphi_i \in \mathrm{PSH}(X, \theta_i)$ so that $T_i = \theta_i + \mathrm{dd^c}\varphi_i$ for each $i = 1, \dots, n$. Fix a reference Kähler form ω on X.

2.1. **The different definitions.** For each $i = 1, \dots, n$, let $(\varphi_i^j)_j$ be a quasi-equisingular approximation of φ_i. Let $(\epsilon_j)_j$ be a decreasing sequence in $\mathbb{R}_{\geq 0}$ with limit 0 so that

$$\varphi_i^j \in \mathrm{PSH}(X, \theta + \epsilon_j \omega)$$

for each $i = 1, \dots, n$ and $j > 0$.

Definition 2.1. The mixed volume of $T_1, \dots, T_n$ in the sense of Cao is defined as follows:

$$\langle T_1, \dots, T_n \rangle_C := \lim_{j \to \infty} \int_X (\theta_1 + \epsilon_j \omega + \mathrm{dd^c}\varphi_1^j) \wedge \dots \wedge (\theta_n + \epsilon_j \omega + \mathrm{dd^c}\varphi_n^j).$$

Here the product is understood in the non-pluripolar sense.

It is shown in [Cao14] Section 2 that this definition is independent of the choices of the θ_i's, the ϵ_j's, the φ_i's, φ_i^j's and ω.

A different definition relies on the $\mathcal{I}$-envelope technique studied in [DX21; DX22]. Recall that the pure volume of a current is defined in [Xia, Definition 3.2.3]:

$$\mathrm{vol}(\theta + \mathrm{dd^c}\varphi) = \int_X (\theta + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}})^n .$$

Definition 2.2. Assume that $\mathrm{vol}\, T_i > 0$ for all $i = 1, \dots, n$. The mixed volume of $T_1, \dots, T_n$ in the sense of Darvas–Xia is defined as follows:

$$\mathrm{vol}(T_1, \dots, T_n) = \int_X (\theta_1 + \mathrm{dd^c} P_{\theta_1}[\varphi_1]_{\mathcal{I}}) \wedge \dots \wedge (\theta_n + \mathrm{dd^c} P_{\theta_n}[\varphi_n]_{\mathcal{I}}) . \tag{2.1}$$

In general, define

$$\mathrm{vol}(T_1, \dots, T_n) = \lim_{\epsilon \to 0+} \mathrm{vol}(T_1 + \epsilon\omega, \dots, T_n + \epsilon\omega). \tag{2.2}$$

This definition is again independent of the choices of ω, the θ_i's and the φ_i's, using the same proof as [Xia, Proposition 3.2.7].

Remark 2.3. When $\mathrm{vol}\, T_i > 0$ for all $i = 1, \dots, n$, the definition (2.2) is compatible with (2.1), thanks to [Xia, Example 7.1.2].

When $T_1 = \cdots = T_n = T$, the above definition is compatible with pure case:

Proposition 2.4. *We always have*

$$\operatorname{vol}(T, \dots, T) = \operatorname{vol} T.$$

Proof. Write $T = \theta_\varphi$. In more concrete terms, we need to show that

$$\lim_{\epsilon \to 0+} \int_X (\theta + \epsilon\omega + \mathrm{dd^c} P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}})^n = \int_X (\theta + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}})^n.$$

We may replace φ by $P_\theta[\varphi]_{\mathcal{I}}$ and assume that φ is $\mathcal{I}$-model in $\operatorname{PSH}(X, \theta)$. Then we claim that

$$\varphi = \inf_{\epsilon > 0} P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}.$$

From this, our assertion follows from [Xia, Proposition 3.1.9].

The $\leq$ direction is clear. For the converse, it suffices to show that for each prime divisor E over X, we have

$$\nu(\varphi, E) \leq \nu\left(\inf_{\epsilon>0} P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}, E\right).$$

We simply compute

$$\nu\left(\inf_{\epsilon>0} P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}, E\right) \geq \sup_{\epsilon>0} \nu\left(P_{\theta+\epsilon\omega}[\varphi]_{\mathcal{I}}, E\right) = \nu(\varphi, E).$$

□

Proposition 2.5. *Both volumes are symmetric. The mixed volume in the sense of Cao is multi-$\mathbb{Q}_{\geq 0}$-linear, while the mixed volume in the sense of Darvas–Xia is multi-$\mathbb{R}_{\geq 0}$-linear.*

The multi-$\mathbb{Q}_{\geq 0}$-linearity means two things:

(1) For each $\lambda \in \mathbb{Q}_{\geq 0}$, we have

$$\langle \lambda T_1, T_2, \dots, T_n \rangle_C = \lambda \langle T_1, T_2, \dots, T_n \rangle_C.$$

(2) If T_1' is anther closed positive $(1,1)$-current, then

$$\langle T_1 + T_1', T_2, \dots, T_n \rangle_C = \langle T_1, T_2, \dots, T_n \rangle_C + \langle T_1', T_2, \dots, T_n \rangle_C. \tag{2.3}$$

Multi-$\mathbb{R}_{\geq 0}$-linearity is defined similarly.

Proof. We first handle the mixed volumes in the sense of Cao. Only the property (2.3) needs a proof. But this follows from the fact that the sum of two quasi-equisingular approximations is again a quasi-equisingular approximation. See [Xia, Theorem 6.2.2, Corollary 7.1.2].

Next we handle the case of mixed volumes in the sense of Darvas–Xia. We only need to show that

$$\operatorname{vol}(T_1 + T_1', T_2, \dots, T_n) = \operatorname{vol}(T_1, T_2, \dots, T_n) + \operatorname{vol}(T_1', T_2, \dots, T_n).$$

Thanks to the definition (2.2), we may assume that $\operatorname{vol} T_i > 0$ for each i and $\operatorname{vol} T_1' > 0$. Write $T_1' = \theta_1' + \mathrm{dd^c}\varphi_1'$. Then

$$P_{\theta_1}[\varphi_1]_{\mathcal{I}} + P_{\theta_1'}[\varphi_1']_{\mathcal{I}} \sim_P P_{\theta_1+\theta_1'}[\varphi_1 + \varphi_1']_{\mathcal{I}}$$

as a consequence of [Xia, Example 7.1.2, Proposition 7.2.1]. Our assertion follows. □

2.2. The equivalence.

Theorem 2.6. *We have*

$$\langle T_1, \dots, T_n \rangle_C = \operatorname{vol}(T_1, \dots, T_n). \tag{2.4}$$

Proof. **Step 1**. We reduce to the case where $T_1 = \cdots = T_n$.

Suppose this special case has been proved. Let $\lambda_1, \dots, \lambda_n \in \mathbb{Q}_{>0}$ be some numbers. Then

$$\langle \sum_{i=1}^n \lambda_i T_i, \dots, \sum_{i=1}^n \lambda_i T_i \rangle_C = \operatorname{vol}(\sum_{i=1}^n \lambda_i T_i).$$

It follows from Proposition 2.5 that both sides are polynomials in the λ_i's. Comparing the coefficients of $\lambda_1 \cdots \lambda_n$, we conclude (2.4).

From now on, we assume that $T_1 = \cdots = T_n = T$. Write $T = \theta_\varphi$.

Step 2. We reduce to the case where T is a Kähler current. For this purpose, it suffices to show that

$$\lim_{\epsilon \to 0+} \langle T_1 + \epsilon\omega, \dots, T_n + \epsilon\omega \rangle_C = \langle T_1, \dots, T_n \rangle_C,$$

which is obvious by definition.

Step 3. Let $(\varphi^j)_j$ be a quasi-equisingular approximation of φ in $\mathrm{PSH}(X,\theta)$. We need to show that

$$\lim_{j\to\infty} \int_X (\theta + \mathrm{dd^c}\varphi^j)^n = \int_X (\theta + \mathrm{dd^c} P_\theta[\varphi]_{\mathcal{I}})^n.$$

This follows from [Xia, Corollary 7.1.2].

□

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