NOTE ON PARTIAL OKOUNKOV BODIES — THE POINT OF VIEW OF B-DIVISORS

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1. Introduction

In this note, I will explain a more algebraic point view to the partial Okounkov bodies introduced in [Xia21].

The main theorem is the following:

thm:main **Theorem 1.1.** *Let X be an irreducible complex projective manifold of dimension n and (L,ϕ) be a Hermitian pseudo-effective line bundle on X with positive volume. Fix a valuation $\nu\colon \mathbb{C}(X)^\times \to \mathbb{Z}^n$ of rank n*[*].

Then the partial Okounkov body $\Delta_\nu(L,\phi)$ admits the following expression:

{eq:main}
$$\Delta_\nu(L,\phi) = \nu(\phi) + \lim_{\pi\colon Z\to X} \Delta_\nu\left(c_1(\pi^*L) - \{\mathrm{Sing}_Z(\phi)\}\right), \tag{1.1}$$

where π runs over all smooth birational modifications of X.

This theorem needs some explanations. Here $\mathrm{Sing}_Z(\phi)$ denotes the divisorial part of the Siu decomposition of $\mathrm{dd}^{\mathrm{c}}\pi^*\phi$. The notation $\{\bullet\}$ means the associated numerical class. The limit is the Hausdorff limit. The valuation $\nu(\phi)$ is defined in [DX24] using the trace operator.

This theorem shows that the partial Okounkov bodies admit a natural interpretation in terms of the associated b-divisors.

One could easily generalize the argument below to the transcendental case, but I find no applications of such generalizations, so I will content myself to the algebraic setting.

One remark: in [DX24], we only explained how to define the trace operator when the subvariety is smooth. In general, the trace operator gives a P-equivalence class on the normalization of the subvariety. We will use these results freely.

2. Some preliminaries

ationincseq **Lemma 2.1.** *Let E_j be a countable family of distinct prime divisors on X. Consider $a_{ij} \in \mathbb{R}_{\geq 0}$ for all $i,j>0$. We assume that the sequence (a_{ij}) for fixed j is increasing in i. Moreover, assume that $a_j \coloneqq \lim_{i\to\infty} a_{ij} < \infty$. Assume that the series $\sum_j a_j[E_j]$ converges, then*

$$\lim_{i\to\infty} \nu\left(\sum_j a_{ij}[E_j]\right) = \nu\left(\sum_j a_j[E_j]\right).$$

[*] Here implicitly, we assume that ν is surjective.

Proof. We may assume that the valuation ν is induced by a smooth flag $Y_\bullet$.

We argue by induction on the dimension n. When $n=1$, there is nothing to argue. Assume that $n>1$ and the case $n-1$ is known. We may assume that Y_1 is not among the E_j's. Write μ for the valuation on Y_1 induced by the truncated flag. Then we need to prove the following:

$$\lim_{i\to\infty}\mu\left(\sum_j a_{ij}[E_j]|_{Y_1}\right)=\mu\left(\sum_j a_j[E_j]|_{Y_1}\right).$$

Note that $[E_j]|_{Y_1}$ is again the current of integration of an effective divisor on Y_1 (this can be seen using the Lelong–Poincaré formula for example), so the desired convergence follows by induction. □

Lemma 2.2. *Let T be a closed positive $(1,1)$-current on X. Then we have*

$$\lim_{\pi\colon Z\to X}\nu(\mathrm{Sing}_Z(T))=\nu(T). \tag{2.1}$$

Proof. We may assume that ν is induced by a smooth flag $Y_\bullet$ on X.

Given $\pi\colon Z\to X$, we let W_1 denote the strict transform of Y_1 in Z. The restriction $\pi_1\colon W_1\to Y_1$ is necessarily birational.

We will argue by induction. The case $n=1$ is trivial. Assume that $n>1$ and the case $n-1$ is known.

We may clearly assume that $\nu(T,Y_1)=0$. By definition, we have

$$\nu(T)=(0,\mu(\mathrm{Tr}_{Y_1}(T)))\,,$$

where μ denotes the valuation induced by the flag on Y_1 induced by $Y_\bullet$.

Observe that modifications of the form $\pi_1\colon W_1\to Y_1$ is cofinal in the directed set of modifications of Y_1. This is obvious since the modifications given by compositions of blow-ups with smooth centers on Y_1 are cofinal.

Therefore, by induction, it suffices to argue that for any $\pi\colon Z\to X$, we have

$$\nu(\mathrm{Sing}_Z(T))=\Big(0,\mu(\mathrm{Sing}_{\widetilde{W_1}}(\mathrm{Tr}_{Y_1}(T)))\Big)\,, \tag{2.2}$$

where $\widetilde{W_1}$ is the normalization of W_1. Let $\widetilde{\pi_1}$ denote the normalization of π_1 so that we have a commutative diagram

$$\begin{array}{ccccc}
\widetilde{W_1} & \longrightarrow & W_1 & \hookrightarrow & Z\\
\big\downarrow{\scriptstyle\widetilde{\pi_1}} & & \big\downarrow{\scriptstyle\pi_1} & & \big\downarrow{\scriptstyle\pi}\\
Y_1 & = & Y_1 & \hookrightarrow & X.
\end{array}$$

From the birational behaviour of the trace operator proved in [DX24], we know that

$$\widetilde{\pi_1}^*\,\mathrm{Tr}_{Y_1}(T)\sim_P \mathrm{Tr}_{W_1}(\pi^*T).$$

So we only need to prove

$$\nu(\mathrm{Sing}_Z(\pi^*T))=\Big(0,\mu(\mathrm{Sing}_{\widetilde{W_1}}(\mathrm{Tr}_{W_1}(\pi^*T))\Big)\,,$$

In order to prove this, we may add a Kähler form to T and assume that T is a Kähler current. Take a quasi-equisingular approximation T_j of T. Then π^*T_j is a quasi-equisingular approximation of π^*T. By the decreasing continuity of the trace operator proved in [DX24], the d_S-continuity of Lelong numbers proved in [Xia22] and Lemma 2.1, both sides are continuous along quasi-equisingular approximations, we reduce to the case where π^*T has analytic singularities. In this case, take a suitable resolution and argue as before, we may assume that $\pi^*T=[D]$ for a snc $\mathbb{Q}$-divisor D. By additivity, we finally reduce to the case where D is a prime divisor on X different from Y_1. The problem is reduced to

$$\nu([D])=(0,\mu([D]|_{Y_1})),$$

which is clear by definition. □

3. The proof

Now let us begin the argument of Theorem 1.1. We argue by induction on n. The case $n = 1$ is of course trivial. Let us assume that $n > 1$ and the result is known in dimension $n - 1$.

We first make a few simplifications. Observe that (1.1) is birationally invariant, so we may assume that ν is equivalent the the valuation induced by a smooth flag. Furthermore, we reduce to the case that ν is the valuation induced by a smooth flag $Y_\bullet$.

It would be more convenient to use the language of currents. We shall write $T = \mathrm{dd^c}\phi$. Then one needs to prove two things: first of all, the limit in (1.1) exists; secondly,

$$\Delta_\nu(T) = \nu(T) + \lim_{\pi\colon Z\to X} \Delta_\nu(c_1(\pi^*L) - \{\mathrm{Sing}_Z(T)\}). \tag{3.1}$$

We may replace T by $T - \nu(T, Y_1)[Y_1]$ and L by the numerical class $L - \nu(T, Y_1)[Y_1]$, so that we may reduce to the case where $\nu(T, Y_1) = 0$. But now L is replaced by a big numerical class α on X in the real Néron–Severi group of X. By perturbation, we may assume α lies in the rational Néron–Severi group. After a rescaling, we reduce back to the case where α is represented by a line bundle L.

Eventually we want to show (3.1) assuming that $\nu(T, Y_1) = 0$.

Let us prove (3.1). As shown in [DX24], we have

$$\Delta_\nu\left(c_1(\pi^*L) - \{\mathrm{Sing}_Z(T)\}\right) = \overline{\{\nu(S) : S \in c_1(\pi^*L) - \{\mathrm{Sing}_Z(T)\}\}}.$$

Therefore,

$$\Delta_\nu\left(c_1(\pi^*L) - \{\mathrm{Sing}_Z(T)\}\right) + \nu(\mathrm{Sing}_Z(T)) \subseteq \overline{\{\nu(S) : S \in c_1(L), \pi^*S \geq \mathrm{Sing}_Z(T)\}}.$$

We observe that the right-hand side is decreasing with respect to π, which together with Lemma 2.2 implies that the net of convex bodies $\Delta_\nu(c_1(\pi^*L) - \{\mathrm{Sing}_Z(T)\})$ for various Z is uniformly bounded. Suppose that Δ is the limit of a subnet. Then we have

$$\Delta + \nu(T) \subseteq \overline{\{\nu(S) : S \in c_1(L), S \preceq_{\mathcal{I}} T\}}.$$

As shown in [DX24], the right-hand side is exactly $\Delta_\nu(T)$. So

$$\Delta + \nu(T) \subseteq \Delta_\nu(T).$$

But observe that both sides have the same volume, as computed in [DX24] and [Xia22]. So equality holds.

It follows from the Blaschke selection theorem that the limit in (3.1) exists and (3.1) holds.

References

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