RADIAL CALABI FLOW

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1. Introduction

This is one of my unfinished projects. I began to work on this problem at the end of 2018. The primary goal at that moment was to understand the relation between the set of finite entropy geodesic rays and the set of maximal rays in the sense of [BBJ15]. After a few unsuccessful attempts, I put aside this project for a while. At the beginning of 2020, Chi Li ([Li20]) made the surprising observation that the former set is contained in the latter by elementary methods. The result of Li renders the present project almost useless. However, at the end of 2020, when cleaning my overleaf repository, I realized that this flow may be of interest for other purposes.

I decided to make these notes available to the public, with the hope that someone could finish this project and find further applications.

2. Radial Calabi flow

Let X be a projective manifold of dimension n. Let L be an ample line bundle on X. Fix a Kähler metric ω in $c_1(L)$. Let $V = c_1(L)^n$. Let $\mathcal{R}^2$ be the space of geodesic rays in $\mathcal{E}^2(X,\omega)$ defined in [DL18]. Let $\mathcal{E}^{2,\mathrm{NA}}$ be the set of maximal geodesic rays (in the sense of [BBJ15]) in $\mathcal{R}^2$.

Let $\mathbf{M} : \mathcal{R}^2 \to (-\infty, \infty]$ be the radial Mabuchi K-energy. Let $\mathbf{E} : \mathcal{R}^2 \to \mathbb{R}$ be the radial Monge–Ampère energy.

Theorem 2.1 ([DL18], [Xia19])**.** *$\mathcal{R}^2$ is an Hadamard space and $\mathbf{M}$ is a convex lsc function on $\mathcal{R}^2$.*

We propose the study of the gradient flow of $\mathbf{M}$. This flow can be constructed using the general theory of Hadamard spaces ([AGS08], [Bač14]). The flow will be called either the *radial Calabi flow* or the *Donaldson–Futaki flow* (if you have a better name, please let me know).

Let $\ell^0 \in \mathcal{R}^2$ with $\mathbf{M}(\ell) < \infty$. We write ℓ^s ($s \geq 0$) for the radial Calabi flow with initial value ℓ^0.

Theorem 2.2 ([Li20])**.** *We have*

$$\overline{\mathrm{Dom}(\mathbf{M})} = \mathcal{E}^{2,\mathrm{NA}}.$$

Recall that $\mathrm{Dom}(\mathbf{M}) = \mathbf{M}^{-1}(\mathbb{R})$.

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Strictly speaking, for $\subseteq$ direction, in addition to what is proved in [Li20], we need to know that $\mathcal{E}^{2,\mathrm{NA}}$ is closed in $\mathcal{R}^2$, which follows from the fact that $\mathcal{E}^{1,\mathrm{NA}}$ is complete (Boucksom–Jonsson). The converse follows from the approximation method of [BBJ15].

Corollary 2.3. *The space $\mathcal{E}^{2,\mathrm{NA}}$ is an Hadamard space.*

Most results are established in Theorem 2.1, we only need to show that given two points ℓ^0, ℓ^1 in $\mathcal{E}^{2,\mathrm{NA}}$, all points on the geodesic connecting them are maximal. By Theorem 2.2 and the stability of geodesics, we may assume that $\ell^0, \ell^1 \in \mathrm{Dom}(\mathbf{M})$, then the desired result follows from the convexity of $\mathbf{M}$.

Hence the radial Calabi flow is naturally a flow on $\mathcal{E}^{2,\mathrm{NA}}$.

Conjecture 2.1. *$\ell^{s,\mathrm{NA}}$ is the gradient flow of the M^{NA}-functional with initial value $\ell^{0,\mathrm{NA}}$.*

Here $M^{\mathrm{NA}}(\phi) = \bar{S}E^{\mathrm{NA}}(\phi) + E_R^{\mathrm{NA}}(\phi) + \int_{X^{\mathrm{an}}} A_X \,\mathrm{MA}(\phi)$, where X^{an} is the Berkovich analytification of X with respect to the trivial valuation on $\mathbb{C}$.

In fact, the Hadamard space structure can be seen intrinsically from the NA point of view. To myself, add details later.

For each $\varphi \in \overline{\mathrm{Dom}(M)}$, we let $(S_t\varphi)_{t\geq 0}$ be the weak Calabi flow ([BDL17]) with initial value φ.

Proposition 2.4. *For each $s \geq 0$, $t \geq 0$, let $(\ell_a^{s,t})_{a\in[0,t]}$ be the geodesic segment ℓ_0^0 to $S_{st}\ell_t^0$. For each fixed $a \geq 0$, $\ell_a^{s,t}$ converges to ℓ_a^s in $\mathcal{E}^2$ as $t \to \infty$.*

The proof is a tedious definition-checking.

Corollary 2.5. *The functional $\mathbf{E}$ is constant along the flow ℓ^s. In particular, $E^{\mathrm{NA}}(\ell^{s,\mathrm{NA}})$ is also constant.*

Now we propose the following conjectures.

Conjecture 2.2. *When (X, L) is K-semi-stable, the normalized K-energy*

$$\widetilde{\mathbf{M}}(\ell) := \frac{\mathbf{M}(\ell)}{\|\ell\|_1}$$

is decreasing along the flow.

I am not very certain about this conjecture.

Our final goal is to prove the following:

Conjecture 2.3. *(X, L) is uniformly K-stable iff (X, L) is geodesically stable.*

The same holds for G-equivariant stability.

We also have the following regularity conjecture.

Conjecture 2.4. *We have $\ell^s \in \mathcal{R}^{1,1}$ for any $s > 0$.*

In another direction, by [Xia], one expects that

$$\mathbf{M}(\ell) = \int_{-\infty}^{\infty} M[\hat{\ell}_\tau]\,\mathrm{d}\tau\,,$$

where $\hat{\ell}_\tau = \inf_{t\geq 0} \ell_t - t\tau$ is the Legendre transform of ℓ ([RWN14]), $M[\bullet]$ is a functional of $\mathscr{I}$-model singularities studied in [Xia].

We have to understand the following questions:

How to understand $(\widehat{\ell^s})^{\mathrm{an}}_{\bullet}$?

How to compute d_2 in terms of $\hat{\ell}^1_{\bullet}$ and $\hat{\ell}^0_{\bullet}$?

We hope that there is a meaningful flow driving $(\widehat{\ell^s})^{\mathrm{an}}_{\tau}$ at any fixed τ. This also aligns with the general philosophy of [Xia] and [DX20]. In some sense, we expect the behaviour at different τ to be *partially* decoupled.

This may sound astonishing: Is there ever any known flow of interest running in the space of all (possibly unbounded, with non-zero Lelong numbers etc.) qpsh functions (Archimedean or non-Archimedean)?

References

[AGS08] L. Ambrosio, N. Gigli, and G. Savaré. Gradient flows: in metric spaces and in the space of probability measures. Springer Science & Business Media, 2008.

[Bač14] M. Bačák. Convex analysis and optimization in Hadamard spaces. Vol. 22. Walter de Gruyter GmbH, 2014.

[BBJ15] R. J. Berman, S. Boucksom, and M. Jonsson. A variational approach to the Yau–Tian–Donaldson conjecture. 2015. arXiv: 1509.04561.

[BDL17] R. J. Berman, T. Darvas, and C. Lu. Convexity of the extended K-energy and the large time behavior of the weak Calabi flow. *Geometry & Topology* 21.5 (2017), pp. 2945–2988.

[DL18] T. Darvas and C. H. Lu. Uniform convexity in L^p-Mabuchi geometry, the space of rays, and geodesic stability. 2018. arXiv: 1810.04661.

[DX20] T. Darvas and M. Xia. The closures of test configurations and algebraic singularity types. 2020. arXiv: 2003.04818.

[Li20] C. Li. Geodesic rays and stability in the cscK problem. 2020. arXiv: 2001.01366.

[RWN14] J. Ross and D. Witt Nyström. Analytic test configurations and geodesic rays. *Journal of Symplectic Geometry* 12.1 (2014), pp. 125–169.

[Xia] M. Xia. Pluripotential-theoretic stability thresholds (to appear).

[Xia19] M. Xia. On sharp lower bounds for Calabi type functionals and destabilizing properties of gradient flows. 2019. arXiv: 1901.07889 [math.DG].

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