NOTE ON RELATIVE NORMALISATIONS

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Contents

1. Introduction

We explain some well-known results concerning the relative normalization of schemes and stacks. We will take results in Stacks Project for granted.

We use the abbreviation *qcqs* for quasi-compact quasi-separated.

This notes come from an attempt to understand constructions in [DR73] and [KM85].

2. Relative normalisation of schemes

Let X be a scheme in this section. Recall that for morphism of schemes, integral=affine+universally closed.

Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. Consider the subsheaf $\mathcal{A}'$:

{eq:inteclo}

$$\mathcal{A}'(U) := \{s \in \mathcal{A}(U) : s_x \in \mathcal{A}_x \text{ is integral over } \mathcal{O}_{X,x} \text{ for all } x \in U\} \tag{2.1}$$

for any open subset $U \subseteq X$.

It is easy to see that $\mathcal{A}'$ is a sheaf of quasi-coherent sheaf of $\mathcal{O}_X$-algebras. Moreover, for any affine open U, $\mathcal{A}'(U)$ is the integral closure of $\mathcal{O}_X(U)$ in $\mathcal{A}(U)$. For any $x \in X$, $\mathcal{A}'_x$ is the integral closure of $\mathcal{O}_{X,x}$ in $\mathcal{A}_x$. See [Stacks, Tag 035F].

Definition 2.1. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_X$-algebras. The *integral closure* of $\mathcal{O}_X$ in $\mathcal{A}$ is the quasi-coherent sheaf of $\mathcal{O}_X$-algebras $\mathcal{A}'$ constructed in (2.1).

In terms of schemes, we have

Definition 2.2. Let $f : Y \to X$ be a qcqs morphism of schemes. Let $\mathcal{O}'$ be the integral closure of $\mathcal{O}_X$ in $f_*\mathcal{O}_Y$. The *normalisation* of X in Y (along f) is the morphism

$$\overline{X}^Y := \underline{\operatorname{Spec}}_X \mathcal{O}' \to X\,.$$

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From the universal property of the relative spectrum, we have a natural commutative diagram:

$$\begin{array}{ccc} Y & \longrightarrow & \overline{X}^Y \\ & {\scriptstyle f}\searrow & \swarrow \\ & X & \end{array}$$

It is clear from the definition that $\overline{X}^Y \to X$ is integral.

In practice, it is useful to characterize the normalisation by a universal property:

prop:univ **Proposition 2.3.** *Let $f : Y \to X$ be a qcqs morphism of schemes. Then for any factorization of f into $Y \to Z \to X$ with $Z \to X$ integral, there is a unique morphism $\overline{X}^Y \to Z$ making the following diagram commute:*

$$\begin{array}{ccccc} Y & \longrightarrow & \overline{X}^Y & \longrightarrow & X \\ & \searrow & \vdots & \nearrow {\scriptstyle g} & \\ & & Z & & \end{array}.$$

Proof. As $Z \to X$ is integral, it is affine, so we may identify Z with $\underline{\mathrm{Spec}}_X\, g_*\mathcal{O}_Z$. Then morphism $Y \to Z$ over X is then identified with an $\mathcal{O}_X$-linear homomorphism $g_*\mathcal{O}_Z \to f_*\mathcal{O}_Y$. As g is integral, the image of this homomorphism is in $\mathcal{O}'$, the integral closure of $\mathcal{O}_X$ in $f_*\mathcal{O}_Y$. So we get a map $\overline{X}^Y \to Z$. This map clearly has the desired properties. □

One can reformulate the universal property in his/her favorite ways: in terms of adjoint functors or the associated Yoneda functors.

From this universal property, it is clear that if $Y \to X$ is already integral, then $\overline{X}^Y \to X$ is nothing but $Y \to X$. From [stacks-project Stacks, Tag 03GQ], more generally, if $f : Y \to X$ is qcqs and universally closed, then the normalization is just $\underline{\mathrm{Spec}}_X\, f_*\mathcal{O}_X$.

Relative normalization behaves well under composition.

Corollary 2.4. *Let $g : Z \to Y$, $f : Y \to X$ be two qcqs morphisms of schemes. Then there is a natural isomorphism $\overline{X}^Z \cong \overline{\overline{X}^Y}^Z$.*

Proof. We have the following commutative diagram

$$\begin{array}{ccccc} Z & \longrightarrow & & \longrightarrow & \overline{X}^Z \\ \Big\downarrow {\scriptstyle g} & \searrow & & & \\ & & \overline{\overline{X}^Y}^Z & & \\ & & \Big\downarrow & & \\ Y & \longrightarrow & \overline{X}^Y & & \\ \Big\downarrow {\scriptstyle f} & \swarrow & & & \\ X & \longleftarrow & & \longleftarrow & \overline{X}^Z \end{array}.$$

From Proposition 2.3, we can easily construct a morphism $\overline{X}^Z \to \overline{\overline{X}^Y}^Z$ making the diagram commute.

Conversely, we can successively construct $\bar{X}^Z \to \bar{X}^Y$ from the factorization $Z \to \overline{X}^Y \to X$ and then $\overline{\overline{X}^Y}^Z \to \overline{X}^Z$ from the factorization $Z \to \overline{X}^Z \to \overline{X}^Y$. By definition, the map also makes the diagram commute. It is easy to see that these maps are inverse to each other. □

The universal property also allows us to make sense of other functorial constructions. Here is an example.

Corollary 2.5. *Let*

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X' & \longrightarrow & X \end{array}$$

be a commutative diagram of schemes with f and f' qcqs. Then there is a natural morphism $\overline{X'}^{Y'} \to \overline{X}^Y$ making the following diagram commute:

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \overline{X'}^{Y'} & \longrightarrow & \overline{X}^Y \\ \downarrow & & \downarrow \\ X' & \longrightarrow & X \end{array} .$$

See [Stacks, Tag 035J].

It is immediately clear from the definition that the construction of $\overline{X}^Y$ is Zariski local on X. In fact, we have more

thm:smoothbc **Theorem 2.6.** *The construction of the integral closure commutes with smooth base change. To be more precise, let*

$$\begin{array}{ccc} Y' & \longrightarrow & Y \\ \downarrow{\scriptstyle f'} & \square & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array}$$

be a Cartesian square of schemes with $g : X' \to X$ smooth and $Y \to X$ qcqs, then the natural morphism

$$\overline{X'}^{Y'} \to \overline{X}^Y \times_X X'$$

is an isomorphism.

This is the globalization of the fact that the relative integral closure of rings commutes with smooth base change. See [Stacks, Tag 03GV].

Proof. The natural morphism can be constructed by the universal property. Let $\mathcal{A}'$ be the integral closure of $\mathcal{O}_X$ in $f_*\mathcal{O}_Y$. Then $\overline{X}^Y \times_X X' = \underline{\mathrm{Spec}}_{X'}\, g^*\mathcal{A}'$. By [Stacks, Tag 03GG], $g^*\mathcal{A}'$ is the integral closure of $\mathcal{O}_{X'}$ in $g^*f_*\mathcal{O}_Y$ (here is where we need g to be smooth). As g is flat, $g^*f_*\mathcal{O}_Y$ is nothing but $f'_*\mathcal{O}_{Y'}$, we conclude. □

Next we recall a few persistence results under normalization.

prop:red **Proposition 2.7.** *Let $f : Y \to X$ be a qcqs morphism of schemes. Assume that Y is reduced then so is $\overline{X}^Y$.*

This is immediately clear from definition.

The normalization is useful due to Zariski's main theorem.

thm:ZMTscheme **Theorem 2.8** (Zariski's main theorem)**.** *Assume that $f : Y \to X$ is a quasi-finite and separated morphism of schemes. Then the natural morphism $Y \to \overline{X}^Y$ is a quasi-compact open immersion.*

If we assume moreover that X is Nagata, f is of finite type and Y is reduced, then $\overline{X}^Y \to X$ is finite.

See [Stacks, Tag 02LR] and [Stacks, Tag 03GR]. See [Stacks, Tag 033S] for the notion of Nagata schemes. Schemes essentially of finite type over a field or $\mathbb{Z}$ are both examples of Nagata schemes.

So integral closure provides a compactification of quasi-finite morphisms under mild assumptions. A very common situation that happens in reality is as follows: $f : Y \to U$ is a finite morphism and $U \subseteq X$ is an open subset of a Nagata scheme X. Assume that Y is reduced. Then $Y \to \overline{X}^Y$ is an open immersion and $\overline{X}^Y \to X$ is finite. Moreover, as the integral closure is Zariski local on the base, we see immediately that the inverse image of U in $\overline{X}^Y \to X$ is exactly $f : Y \to U$. See the commutative diagram

$$\begin{array}{ccc} Y & \longrightarrow & \overline{X}^Y \\ \downarrow{\scriptstyle f} & & \downarrow \\ U & \longrightarrow & X \end{array} .$$

If we assume moreover that U is dense in X, then $\overline{X}^Y \to X$ is the unique finite normal compactification of Y over X. In fact, if $Z \to X$ is another such compactification, then we have a morphism $\overline{X}^Y \to Z$, which restricts to identity on Y. It follows from [EGA IV$_3$, Corollaire 8.12.10] and Proposition 2.7 that this morphism is in fact an isomorphism.

3. Relative normalisation of stacks

We follow [MO20]. In this section, when we talk about Artin stacks/algebraic stacks, we do not assume any separatedness, exactly as in Stacks Project. We fix a base scheme S, all stacks will be stacks over S. To avoid set theoretic issues, we pick the fppf site as in [Stacks, Tag 021L] and denote it by $(\mathrm{Sch}/S)_{fppf}$.

Let $\mathfrak{X}$ be an algebraic stack, which is fixed through this section.

Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_{\mathfrak{X}}$-algebras. Here by sheaves, we mean sheaves on the big fppf site over S. We define a subsheaf $\mathcal{A}'$ by

{eq:inteclo2}
$$\mathcal{A}'(U) := \left\{ s \in \mathcal{A}(U) : s \text{ is integral over } H^0(U, \mathcal{O}_U) \right\} \tag{3.1}$$

for all affine schemes U in $(\mathrm{Sch}/S)_{fppf}$. As the property of generating a finite module can be checked fpqc locally, we find that $\mathcal{A}'$ generates a fppf sheaf, still denoted by $\mathcal{A}'$. Moreover, $\mathcal{A}'$ is a quasi-coherent sheaf of $\mathcal{O}_{\mathfrak{X}}$-algebras.

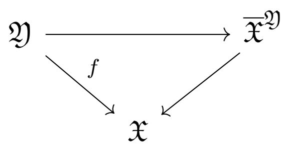
Exactly as in the case of schemes, we have

Definition 3.1. Let $\mathcal{A}$ be a quasi-coherent sheaf of $\mathcal{O}_{\mathfrak{X}}$-algebras. The *integral closure* of $\mathcal{O}_{\mathfrak{X}}$ in $\mathcal{A}$ is the quasi-coherent sheaf of $\mathcal{O}_{\mathfrak{X}}$-algebras $\mathcal{A}'$ constructed in (3.1).

Definition 3.2. Let $f : \mathfrak{Y} \to \mathfrak{X}$ be a qcqs morphism of algebraic stacks. Let $\mathcal{O}'$ be the integral closure of $\mathcal{O}_X$ in $f_*\mathcal{O}_{\mathfrak{Y}}$. The *normalisation* of $\mathfrak{X}$ in $\mathfrak{Y}$ (along f) is the morphism

$$\overline{\mathfrak{X}}^{\mathfrak{Y}} := \underline{\mathrm{Spec}}_{\mathfrak{X}}\, \mathcal{O}' \to \mathfrak{X}\,.$$

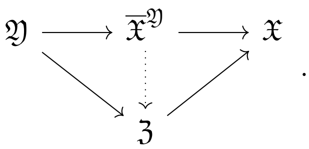
Again, we have the factorization



with $\overline{\mathfrak{X}}^{\mathfrak{Y}} \to \mathfrak{X}$ integral.

With the same proof as Proposition 2.3, we find the universal property of the normalization.

prop:univ2 **Proposition 3.3.** *Let $f : \mathfrak{Y} \to \mathfrak{X}$ be a qcqs morphism of algebraic stacks. Then for any factorization of f into $\mathfrak{Y} \to \mathfrak{Z} \to \mathfrak{X}$ with $\mathfrak{Z} \to \mathfrak{X}$ integral, there is a unique morphism $\overline{\mathfrak{X}}^{\mathfrak{Y}} \to \mathfrak{Z}$ making the following diagram commute:*



So all of the corollaries of the universal property in the previous section can be generalized to algebraic stacks.

The same proof of Theorem 2.6 shows that the integral closure commutes with smooth base change in the current setting as well. Note that in our situation, the cohomology and base change theorem is proved in [LM00, Proposition 13.1.9].

Next, we handle Zariski's main theorem. The things are getting tricky here, because there does not seem to be a good notion of Nagata algebraic stacks, as the property of being Nagata is only local in the smooth topology. So it seems more natural to talk about Nagata stacks only in the Deligne–Mumford case.

thm:ZMTstack **Theorem 3.4** (Zariski's main theorem)**.** *Let $f : \mathfrak{Y} \to \mathfrak{X}$ be a* representable *morphism of algebraic stacks. Assume that f is quasi-finite and separated. Then $\mathfrak{Y} \to \overline{X}^{\mathfrak{Y}}$ is an open immersion.*

See [LM00, Théorème 16.5] for a proof. Note that some of the assumptions in [LM00] are not necessary. Here representable means representable by algebraic spaces, not by schemes.

We state the version for algebraic spaces, which seems to be the most useful case in practice.

Theorem 3.5 (Zariski's main theorem)**.** *Assume that $f : Y \to X$ is a quasi-finite and separated morphism of algebraic spaces. Then the natural morphism $Y \to \overline{X}^Y$ is a quasi-compact open immersion.*

If we assume moreover that X is Nagata, f is of finite type and Y is reduced, then $\overline{X}^Y \to X$ is finite.

Here we say X is Nagata if for one (or equivalent for all) étale chart $Y \to X$ of X, Y is Nagata. See [Stacks, Tag 036E].

This first part of the theorem follows immediately from Theorem 3.4. The second part follows from Theorem 2.8 and descent.

References

DR73 [DR73] P. Deligne and M. Rapoport. Les schémas de modules de courbes elliptiques. *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*. Lecture Notes in Math., Vol. 349. Springer, Berlin, 1973, pp. 143–316.

EGAIV-3 [EGA IV_3] J. Dieudonné and A. Grothendieck. Éléments de géométrie algébrique: IV. Étude locale des schémas et des morphismes de schémas, Troisième partie. Vol. 28. Institut des hautes études scientifiques, 1966, pp. 5–255.

KM85 [KM85] N. M. Katz and B. Mazur. Arithmetic moduli of elliptic curves. Vol. 108. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1985, pp. xiv+514. URL: https://doi.org/10.1515/9781400881710.

LMB00 [LM00] G. Laumon and L. Moret-Bailly. Champs algébriques. Vol. 39. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer-Verlag, Berlin, 2000, pp. xii+208.

MO20 [MO20] L. Meier and V. Ozornova. Rings of modular forms and a splitting of $\text{TMF}_0(7)$. *Selecta Math. (N.S.)* 26.1 (2020), Paper No. 7, 73. URL: https://doi.org/10.1007/s00029-019-0532-5.

stacks-project [Stacks] T. Stacks Project Authors. Stacks Project. http://stacks.math.columbia.edu. 2020.

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