NOTES ON SHIMURA VARIETIES I. BOUNDED SYMMETRIC DOMAINS

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Contents

1. Introduction

This is one of a series of notes prepared for a seminar on the toroidal compactifications of Shimura varieties.

In this note, we explore the proof of the embedding theorem of Borel and Harish-Chandra. Our main reference is the book [Hel79].

Theorem 1.1. *Given any Hermitian symmetric space $D = G/K$ of non-compact type, there are holomorphic open immersions*

$$D \to \mathfrak{p}_+ \to \check{D}\,.$$

Here $\mathfrak{p}_+$ is a complex linear space, the image of D in $\mathfrak{p}_+$ is a bounded symmetric domain. The space $\check{D}$ is the compact dual of D.

The map $D \to \mathfrak{p}_+$ has an explicit formula Corollary 3.20. The image of the embedding $D \to \mathfrak{p}_+$ is characterized by Hermann's convexity theorem.

The embedding $\mathfrak{p}_+ \to \check{D}$ realizes $\mathfrak{p}_+$ as a Zariski open dense subset of $\check{D}$.

A typical example to be kept in mind of these embeddings is

$$\Delta = \{z \in \mathbb{C} : |z| < 1\} \subseteq \mathbb{C} \subseteq \mathbb{P}^1\,.$$

2. The general theory of Hermitian symmetric spaces

We briefly recall the general theory of Hermitian symmetric spaces. For the details, we refer to [Hel79, Section VIII].

Definition 2.1. A Hermitian manifold (D, g) is a *Hermitian symmetric space* if for any $x \in D$, there is a holomorphic geodesic isometry s_x at x: $s_x : D \to D$ is a holomorphic isometry such that x is an isolated fixed point of s_x.

Equivalently, this is equivalent to assuming the existence of s_x at one $x \in D$ and the condition that D is homogeneous with respect to the group of holomorphic isometries.

A Hermitian symmetric space is *irreducible* if it is not isomorphic to the product of two non-trivial Hermitian symmetric spaces.

Theorem 2.2. *A Hermitian symmetric space can be uniquely decomposed as the product of irreducible Hermitian symmetric spaces (up to permuting the factors). An irreducible Hermitian symmetric space falls into one of the following categories:*

1. *Euclidean: if it is isomorphic to $\mathbb{C}^n$ with the flat metric.*
2. *Compact: if it is not Euclidean and has non-negative sectional curvature.*
3. *Non-compact: if it is not Euclidean and has non-positive sectional curvature.*

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Example 2.3. *The following are typical examples:*

1. *Euclidean:* $\mathbb{C}^n$ *with the flat metric.*
2. *Compact:* $\mathbb{P}^1$ *with the usual metric.*
3. *Non-compact:* $\mathbb{H}$ *(the upper half plane) with the Poincaré metric:* $\frac{1}{y^2}\mathrm{d}x \wedge \mathrm{d}y$.

We will say a Hermitian symmetric space is of compact type (resp. non-compact type) if all of its irreducible factors are of compact type (resp. non-compact type).

We will focus on the non-compact type. Let (D, g) be a Hermitian symmetric space of non-compact type. Fix $o \in D$.

Theorem 2.4. *The following are equal*

1. *the identity component of the group of holomorphic isometries of* (M, g).
2. *the identity component of the isometry group of* (M, g) *as a Riemannian manifold;*
3. *the identity component of the group of biholomorphic isomorphisms of* M.

Moreover, this group is adjoint semi-simple [*].

We write G for this group. Write K for the isotropy group of $o \in D$ under the G-action.

Theorem 2.5. *K is a maximal compact subgroup. We have a diffeomorphism $G/K \approx D$. The isometry s_o at o is a Cartan involution of G.*

Assume that D is irreducible, then G is simple and the center of K is isomorphism to $U(1)$.

Assume that D is irreducible. Geometrically, K are holomorphic isomorphisms of D fixing the base point $o \in D$. An element $z \in U(1) = Z(K)$ in the center of K corresponds to the holomorphic isometries of D that fixes o and acts as multiplication by z in the tangent space.

More generally, when D is not necessarily irreducible, we can always find a map $u : U(1) \to G$ sending $z \in U(1)$ to $u(z) \in G$ fixing o and acts as multiplication by z on the tangent space of D at o. This map will be called *Deligne's map*. Observe that $u(-1) = s_o$.

3. The embedding theorems

In this section, we give some details about the structure of Hermitian symmetric spaces following Helgason.

Let (D, g) be a Hermitian symmetric space of non-compact type. Fix a base point $o \in D$. Recall that $G = \mathrm{Iso}(D, g)^+$ is an adjoint semi-simple real Lie group. Since G is adjoint, there is a unique algebraic $\mathbb{R}$-group $\mathcal{G}$ such that

$$\mathcal{G}(\mathbb{R})^+ = \mathrm{Iso}(D, g)^+ .$$

3.1. Cartan decomposition and root decomposition.

Let $\mathfrak{g}$ be the Lie algebra of G. Write $K = K_o$ the isotropy group of o so that $D = G/K$. The Lie algebra of K is denoted by $\mathfrak{k}$. The geodesic symmetry at o induces a Cartan involution σ on $\mathfrak{g}$. In particular, we get the associated Cartan involution:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} .$$

Recall that by definition, this means that $\mathfrak{k}$ is the $+1$-eigenspace of σ and $\mathfrak{p}$ is the -1-eigenspace. [†]

Let $\mathfrak{c} = Z(\mathfrak{k})$ denote the center of $\mathfrak{k}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{k}$. Observe that

$$\mathfrak{c} \subseteq \mathfrak{h}$$

since $\mathfrak{h}$ is self-normalizing. Observe that $\mathfrak{c}$ contains the image of $\mathrm{d}u$ hence non-trivial. In fact, $\mathfrak{c}$ is equal to the image of $\mathrm{d}u$ by [Hel79, Proposition VIII.6.2] if D is irreducible.

We consider the compact dual

$$\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$$

[*] An adjoint semi-simple Lie group is a semi-simple Lie group which has trivial center

[†] Intuitively, $\mathfrak{k}$ corresponds to infinitesimal holomorphic isometries of D fixing o and $p \in \mathfrak{p} = T_o D$ corresponds to an infinitesimal holomorphic isometry of D that translates D in the p direction.

of $\mathfrak{g}$. Let τ be the conjugation of $\mathfrak{g}_{\mathbb{C}}$ induced by $\mathfrak{u}$. The form $B_\tau : \mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathbb{R}$ defined by

$$B_\tau(X,Y) = -B(X,\tau Y)$$

is strictly positive definite. Here B denotes the Killing form of $\mathfrak{g}_{\mathbb{C}}$.

Lemma 3.1. $\mathfrak{h}$ *is also a Cartan subalgebra of* $\mathfrak{g}$.

Proof. We may assume that D is irreducible.

It suffices to show that $\mathfrak{h}$ is self-normalizing.

Observe that $Z_{\mathfrak{g}}(\mathfrak{c}) \supseteq \mathfrak{k}$ by definition of $\mathfrak{c}$. But we know that $Z_{\mathfrak{g}}(\mathfrak{c}) \neq \mathfrak{g}$ as otherwise $Z(\mathfrak{g}) = \mathfrak{c}$ will be non-trivial. But we have

$$Z_{\mathfrak{g}}(\mathfrak{c}) \supseteq \mathfrak{k}\,.$$

Hence equality holds by the maximality of $\mathfrak{k}$ among proper Lie subalgebras of $\mathfrak{g}$ [Hel79, Proposition VIII.5.1].

In particular, $\mathfrak{h}$ is maximal abelian. It remains to show that each element x in $\mathfrak{h}$ is diagonalizable in the adjoint representation of $\mathfrak{g}$. But this follows from the fact that $\operatorname{ad} x$ is skew-symmetric with respect to B_τ. □

Write Δ for the set of non-zero roots of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{h}_{\mathbb{C}}$. In other words, we have the root space decomposition

{eq:rootgh}

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Delta} \mathfrak{g}^\alpha\,. \tag{3.1}$$

Observe that the decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$$

is preserved by $\mathfrak{h}_{\mathbb{C}}$. It follows that each root α is either *compact* in the sense $\mathfrak{g}^\alpha \subseteq \mathfrak{k}_{\mathbb{C}}$ or *non-compact* in the sense $\mathfrak{g}^\alpha \subseteq \mathfrak{p}_{\mathbb{C}}$.

Definition 3.2. Let $W \subseteq V$ be two finite dimensional $\mathbb{R}$-vector spaces. Two given linear ordering on $V^\vee$ and $W^\vee$ are said to be *compatible* if any element in $V^\vee$ that restricts to a positive element in $W^\vee$ is positive.

It is easy to choose compatible linear orderings.

Suppose that we introduce compatible orderings on $i\mathfrak{c}^\vee$ and $i\mathfrak{h}^\vee$. Recall that all roots in Δ are real-valued on $i\mathfrak{h}$ by [Hel79, Lemma VI.3.1], so we get a notion of positive roots Δ^+. Write Q_+ for the positive roots that do not restrict to 0 on $i\mathfrak{c}$. Write

$$\mathfrak{p}_+ = \bigoplus_{\beta\in Q_+} \mathfrak{g}^\beta\,, \quad \mathfrak{p}_- = \bigoplus_{\beta\in Q_+} \mathfrak{g}^{-\beta}\,.$$

Proposition 3.3. *The spaces* $\mathfrak{p}_+$ *and* $\mathfrak{p}_-$ *are both abelian subalgebras of* $\mathfrak{g}_{\mathbb{C}}$ *and*

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_- \oplus \mathfrak{p}_+\,.$$

Moreover, both $\mathfrak{p}_-$ *and* $\mathfrak{p}_+$ *are invariant under* $\operatorname{ad}\mathfrak{k}_{\mathbb{C}}$. *Moreover,*

$$[\mathfrak{p}_+,\mathfrak{p}_-] = \mathfrak{k}_{\mathbb{C}}\,.$$

Proof. Consider a compact root $\alpha \in \Delta$, then $[\mathfrak{c},\mathfrak{g}^\alpha] = 0$ by definition of $\mathfrak{c}$. It follows that $\alpha|_{\mathfrak{c}_{\mathbb{C}}} = 0$. It follows that

$$\mathfrak{p} \supseteq \mathfrak{p}_- + \mathfrak{p}_+\,.$$

Observe that $\alpha+\beta$ is clearly positive (negative) when restricted to $i\mathfrak{c}$ where α is a compact root and $\pm\beta \in Q_+$. So when α is compact, g^α preserves $\mathfrak{p}_\pm$. It follows that $\mathfrak{p}_\pm$ are both preserved by $\operatorname{ad}\mathfrak{k}_{\mathbb{C}}$.

To see that $\mathfrak{p}_+$ is abelian, take $\beta,\gamma \in Q_+$. We need to show that

$$[\mathfrak{g}^\beta,\mathfrak{g}^\gamma] = 0\,.$$

There is nothing to prove if $\beta+\gamma$ is not a root. Otherwise, it is in Q_+. But we know that

$$[\mathfrak{g}^\beta,\mathfrak{g}^\gamma] \subseteq [\mathfrak{p}_+,\mathfrak{p}_+] \subseteq \mathfrak{k}_{\mathbb{C}}\,.$$

It follows that $[\mathfrak{g}^\beta,\mathfrak{g}^\gamma] = 0$. A similar argument shows that $\mathfrak{p}_-$ is also abelian.

Finally, we need to show that $\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_- + \mathfrak{p}_+$. We may assume that D is irreducible. We write

$$\mathfrak{g}_+ = \mathfrak{p}_- + \mathfrak{p}_+ + [\mathfrak{p}_+, \mathfrak{p}_-]\,.$$

We claim that $\mathfrak{g}_+$ is an ideal in $\mathfrak{g}_{\mathbb{C}}$. But recall that $\mathfrak{g}_{\mathbb{C}}$ is a simple Lie algebra, which implies that $\mathfrak{g}_+$ is either 0 or $\mathfrak{g}_{\mathbb{C}}$. The former is clearly impossible. Thus

$$\mathfrak{p}_- + \mathfrak{p}_+ + [\mathfrak{p}_+, \mathfrak{p}_-] = \mathfrak{g}_{\mathbb{C}}\,.$$

Observe that $[\mathfrak{p}_+, \mathfrak{p}_-] \subseteq \mathfrak{k}_{\mathbb{C}}$, our desired result thus follows.

It remains to prove the claim. We write $\mathfrak{q}$ for the orthogonal complement of $\mathfrak{p}_- + \mathfrak{p}_+$ in $\mathfrak{p}_{\mathbb{C}}$ with respect to B_τ. Observe that it suffices to show that

$$[\mathfrak{p}_+, \mathfrak{q}] = [\mathfrak{p}_-, \mathfrak{q}] = 0\,.$$

We only prove the former, as the latter can be proved similarly. Take $T \in \mathfrak{k}_{\mathbb{C}}$, $X \in \mathfrak{p}_+$ and $Y \in \mathfrak{q}$. But we have $\tau T \in \mathfrak{k}_{\mathbb{C}}$ and $\tau[X, \tau T] \in \mathfrak{p}_-$. Thus

$$B_\tau([X, Y], T) = -B_\tau(Y, \tau[X, \tau T]) = 0\,.$$

It follows that $[\mathfrak{p}_+, \mathfrak{q}] = 0$. □

Corollary 3.4. *A root $\alpha \in \Delta$ is compact iff its restriction to $\mathrm{i}\mathfrak{c}$ vanishes.*

Remark 3.5. In fact, we can make the choice of positive roots more canonical. By choosing the ordering corresponding to the Weyl chamber $\mathrm{d}u(\mathrm{i}) \in \mathrm{i}\mathfrak{h}$, we may assume that all roots in Q_+ are positive and all roots in Q_- are negative.

3.2. **Maximal split torus.** Next we want to choose a special maximal abelian subspace of $\mathfrak{p}$.

If $Q \subseteq Q_+$ is non-empty, let

$$\mathfrak{p}_Q = \sum_{\gamma \in Q} (\mathfrak{g}^\gamma + \mathfrak{g}^{-\gamma})\,.$$

Let β be the lowest root in Q and let $Q(\beta)$ denote the set of all $\gamma \in Q$ not equal to β while strongly orthogonal to β (in the sense that $\gamma \pm \beta \notin \Delta$). For each $\alpha \in \Delta$, choose a non-zero vector $X_\alpha \in \mathfrak{g}^\alpha$. Let s be the $\mathbb{R}$-rank of $\mathcal{G}$, namely, the dimension of any maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Recall that s is equal to the rank of the maximal split torus in $\mathcal{G}$.

lizerXbpXmb **Lemma 3.6.** *We have*

$$Z_{\mathfrak{p}_Q}(X_\beta + X_{-\beta}) = \mathbb{C}(X_\beta + X_{-\beta}) + \mathfrak{p}_{Q(\beta)}\,.$$

Proof. We only have to prove the $\subseteq$ direction.

Let $X \in \mathfrak{p}_Q$ and let $Q' = Q \setminus \{\beta\}$. Write

$$X = c_\beta X_\beta + c_{-\beta} X_{-\beta} + \sum_{\gamma \in Q'} \left(c_\gamma X_\gamma + c_{-\gamma} X_{-\gamma}\right)\,.$$

We compute $[X, X_\beta + X_{-\beta}]$. Under the root decomposition (3.1) eq:rootgh, the $\mathfrak{h}_{\mathbb{C}}$ component is given by

$$(c_\beta - c_{-\beta})[X_\beta, X_{-\beta}]\,.$$

Now we assume further that $X \in Z_{\mathfrak{p}_Q}(X_\beta + X_{-\beta})$. This implies that $c_\beta = c_{-\beta}$ and hence

$$\sum_{\gamma \in Q'} \left(c_\gamma X_\gamma + c_{-\gamma} X_{-\gamma}\right)$$

commutes with $X_\beta + X_{-\beta}$. Using the fact that $\mathfrak{p}_\pm$ are both commutative, we find

$$\left[\sum_{\gamma \in Q'} \left(c_\gamma X_\gamma + c_{-\gamma} X_{-\gamma}\right), X_\beta + X_{-\beta}\right] = \sum_{\gamma \in Q'} \left(c_\gamma [X_\gamma, X_{-\beta}] + c_{-\gamma}[X_{-\gamma}, X_\beta]\right) = 0\,.$$

We claim that this implies that

$$c_\gamma [X_\gamma, X_{-\beta}] = 0\,, \quad c_{-\gamma}[X_{-\gamma}, X_\beta] = 0$$

for all $\gamma \in Q'$. It follows that $\sum_{\gamma \in Q'} \left(c_\gamma X_\gamma + c_{-\gamma} X_{-\gamma}\right) \in \mathfrak{p}_Q$ and we conclude.

It remains to prove the claim. Assume otherwise, for example $c_\gamma[X_\gamma, X_{-\beta}] \neq 0$. It follows that $\gamma - \beta \in \Delta$. There has to be soe $\delta \in Q'$ so that

$$c_\gamma[X_\gamma, X_{-\beta}] + c_{-\delta}[X_{-\delta}, X_\beta] = 0\,.$$

Then $\alpha := \gamma - \beta = -\delta + \beta \in \Delta$ and $\gamma = \alpha + \beta$, $\delta = \beta - \alpha$. This contradicts the fact that β is the lowest root in Q. □

Proposition 3.7. *There is a subset $\gamma_1, \dots, \gamma_s$ of Q_+ consisting of pairwise strongly orthogonal roots. In particular,*

$$\mathfrak{a}_{\mathbb{C}} = \sum_{i=1}^{s} \mathbb{C}\,(X_{\gamma_i} + X_{-\gamma_i})$$

is a maximal abelian subspace of $\mathfrak{p}_{\mathbb{C}}$.

Proof. We define a sequence of spaces

$$\mathfrak{p}_{\mathbb{C}} = \mathfrak{p}_1 \supseteq \cdots \supseteq \mathfrak{p}_{s+1} = 0\,,$$

each of the form $\mathfrak{p}_i = \mathfrak{p}_{Q_i}$ inductively. For Q_1, we take $Q_1 = Q_+$ and let γ_1 be the lowest positive root in Q_1. Take

$$\mathfrak{p}_2 = Z_{\mathfrak{p}_1}(\mathfrak{g}^{\gamma_1} + \mathfrak{g}^{-\gamma_1})\,.$$

Take $Q_2 = Q_1(\gamma_1)$. Denote by γ_2 the lowest positive root in Q_2 and continue. In the end, we get the desired sequence $\gamma_1, \dots, \gamma_s$. It remains to prove that if $X \in \mathfrak{p}$ commutes with $\mathfrak{a}_{\mathbb{C}}$ then $X \in \mathfrak{a}_{\mathbb{C}}$. If not, there is $1 \leq r \leq s$ so that $X \in \mathfrak{p}_r + \mathfrak{a}_{\mathbb{C}} \setminus \mathfrak{p}_{r+1} + \mathfrak{a}_{\mathbb{C}}$. We write $X = Y + Z$ for $Y \in \mathfrak{p}_r$ and $Z \in \mathfrak{a}_{\mathbb{C}}$. As both X and Z commute with $X_{\gamma_r} + X_{-\gamma_r}$, so is Y. By the Lemma 3.6, $Y = c(X_{\gamma_r} + X_{-\gamma_r}) + Y_1$ for some $c \in \mathbb{C}$ and $Y_1 \in \mathfrak{p}_{r+1}$. But then $Z_1 = Z + c(X_{\gamma_r} + X_{-\gamma_r}) \in \mathfrak{a}_{\mathbb{C}}$ and hence $X = Y_1 + Z_1 \in \mathfrak{p}_{r+1} + \mathfrak{a}_{\mathbb{C}}$, which is a contradiction. □

We want to choose a real form of $\mathfrak{a}_{\mathbb{C}}$ as well.

For each root $\alpha \in \Delta$, choose $H_\alpha \in \mathrm{i}\mathfrak{h}$ as the dual of α as a form on $\mathfrak{h}_{\mathbb{C}}$ relative to the Killing form. Let

$$h_\alpha = \frac{2}{\alpha(H_\alpha)} H_\alpha \in \mathrm{i}\mathfrak{h}\,.$$

For each $\alpha \in \Delta$, choose $X_\alpha \in \mathfrak{g}^\alpha$ so that

$$X_\alpha - X_{-\alpha}, \mathrm{i}(X_\alpha + X_{-\alpha}) \in \mathfrak{u}\,, \quad [X_\alpha, X_{-\alpha}] = h_\alpha\,.$$

These arrangements are possible, see [Hel79, Lemma VI.3.1]. Define

$$\mathfrak{a} = \sum_{i=1}^{s} \mathbb{R}(X_{\gamma_i} + X_{-\gamma_i})\,.$$

Proposition 3.8. *We have $\mathfrak{a} = \mathfrak{a}_{\mathbb{C}} \cap \mathfrak{p}$. In particular, $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$.*

In particular, there is a maximal split torus $\mathcal{A}$ in $\mathcal{G}$ so that $\mathcal{A}(\mathbb{R})^+ = \exp \mathfrak{a}$.

Proof. In fact, $X_{\gamma_i} + X_{-\gamma_i} \in \mathrm{i}\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}} = \mathfrak{p}$. Thus $\mathfrak{a} \subseteq \mathfrak{a}_{\mathbb{C}} \cap \mathfrak{p}$. On the other hand, take $X \in \mathfrak{a}_{\mathbb{C}}$. Assume that $X \in \mathfrak{p}$, then $\tau X = -X$ and hence all coefficients in front of $X_{\gamma_i} + X_{-\gamma_i}$ have to be real. □

For future use, we introduce

$$x_\alpha = X_\alpha + X_{-\alpha} \in \mathfrak{a}\,, \quad y_\alpha = \mathrm{i}(X_\alpha - X_{-\alpha}) \in \mathfrak{u}$$

for $\alpha = \gamma_i$. We also write x_i for x_{γ_i}. The same applies to other notations like y_i, X_i, X_{-i}.

3.3. **Lie groups.** Let $G_{\mathbb{C}}$ be the simply connected complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

Let us write

$$\mathfrak{n}_+ = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}^{\alpha}\,, \quad \mathfrak{n}_- = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}^{-\alpha}\,.$$

Let $N_\pm$ (resp. $P_\pm$) be the analytic groups corresponding to $\mathfrak{n}_\pm$ (resp $\mathfrak{p}_\pm$). Write $K_{\mathbb{C}}$ for the analytic group corresponding to $\mathfrak{k}_{\mathbb{C}}$. All of these groups are considered as subgroups of $G_{\mathbb{C}}$.

Proposition 3.9. *The map* $\exp : \mathfrak{p}_\pm \to P_\pm$ *is a diffeomorphism. Similarly, the map* $\exp : \mathfrak{n}_\pm \to N_\pm$ *is a diffeomorphism.*

Proof. By Engel's theorem, $\operatorname{ad}\mathfrak{n}_+$ is nilpotent. From the general property of nilpotent Lie groups [Hel79, Corollary VI.4.4, Lemma VI.4.5], $\operatorname{ad}\mathfrak{n}_+$ is diffeomorphic to $\operatorname{Ad}N_+$ through the exponential map. It follows that $\exp : \mathfrak{n}_+ \to N_+$ is a diffeomorphism. As $\mathfrak{p}_+ \subseteq \mathfrak{n}_+$, $\exp : \mathfrak{p}_+ \to P_+$ is a diffeomorphism. The $-$ part is similar. □

Lemma 3.10. *The map* $P_- \times K_{\mathbb{C}} \times P_+ \to G_{\mathbb{C}}$ *given by* $(q,k,p) \mapsto qkp$ *is a differmorphism onto an open submanifold of* $G_{\mathbb{C}}$ *containing* G.

Similarly, $P_+ \times K_{\mathbb{C}} \times P_- \to G_{\mathbb{C}}$ *given by* $(q,k,p) \mapsto qkp$ *is also a diffeomorphism.*

We will write $\mathfrak{n}_\pm$ for the sum of $\mathfrak{g}^\alpha$ for all positive (resp. negative) roots in Δ.

Proof. **Step 1**. We prove that $P_-K_{\mathbb{C}} \cap P_+ = \{1\}$. If not, take a non-trivial element in the intersection y. Select $Y \in \mathfrak{p}_+$ with $y = \exp Y$. Observe that $\operatorname{Ad}(y)(\mathfrak{p}_-) \subseteq \mathfrak{p}_-$ as $y \in P_-K_{\mathbb{C}}$. Write $Y = \sum_{\alpha\in Q_+} c_\alpha X_\alpha$. Let β be the lowest root in Q_+ such that $c_\beta \neq 0$. Then

$$[Y, X_{-\beta}] \equiv c_\beta [X_\beta, X_{-\beta}] \mod \mathfrak{n}_+\,.$$

It follows that

$$\operatorname{Ad}(y)(X_{-\beta}) \equiv X_{-\beta} + c_\beta[X_\beta, X_{-\beta}] \mod \mathfrak{n}_+\,.$$

But further modulo $\mathfrak{n}_-$, we find that $\operatorname{Ad}(y)(X_{-\beta})$ has a non-trivial $\mathfrak{k}$-component, contracting the fact that $\operatorname{Ad}(y)\mathfrak{p}_- \subseteq \mathfrak{p}_-$.

Observe that dually, $P_- \cap K_{\mathbb{C}}P_+ = \{1\}$.

Step 2. $P_- \times K_{\mathbb{C}} \times P_+ \to G_{\mathbb{C}}$ is injective. Suppose that

$$q_1k_1p_1 = q_2k_2p_2\,, \quad q_i \in P_-, k_i \in K_{\mathbb{C}}, p_i \in P_+\,.$$

It follows that

$$k_2^{-1}q_2^{-1}q_1k_2k_2^{-1}k_1 = p_2p_1^{-1}\,.$$

From Step 1, $p_2 = p_1$. Similarly, $q_2 = q_1$. Hence $k_2 = k_1$.

By [Hel79, Lemma VI.5.2], the map is also regular. It follows that the image is a submanifold of $G_{\mathbb{C}}$ of correct dimension and hence the image is open.

Step 3. We show that the image contains G. From the general theory of symmetric spaces,

$$G_{\mathbb{C}} = (\exp\mathfrak{p})K_{\mathbb{C}}\,.$$

Take $X \in \mathfrak{p}$ and write $p = \exp(X/2)$. By the complex Iwasawa decomposition, we write $p = uan$ with $u \in U = \exp\mathfrak{u}$, $a \in A^* = \exp i\mathfrak{h}$, $n \in N_+ = \exp\mathfrak{n}_+$. Thus,

$$\tau p = p^{-1} = ua^{-1}\tau(n)\,,$$

so

$$\exp X = p^2 = \tau(n^{-1})a^2n \in N_-A^*N_+ \subseteq P_-K_{\mathbb{C}}P_+\,.$$

□

Lemma 3.11. *The group* $K_{\mathbb{C}}P_+$ *is closed in* $G_{\mathbb{C}}$. *Its Lie algebra is* $\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_+$. *Similarly,* $K_{\mathbb{C}}P_-$ *is closed in* $G_{\mathbb{C}}$ *with Lie algebra* $\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_-$.

Proof. First observe that P_+ is closed in N_+, which follows from the Lie algebra picture under taking log. In particular, P_+ is closed in $G_{\mathbb{C}}$. The group $K_{\mathbb{C}}$ is closed in $G_{\mathbb{C}}$ due to the existence of Cartan decomposition. Now consider a sequence $k_np_n \in K_{\mathbb{C}}P_+$ converging to some element in $G_{\mathbb{C}}$. Applying $\sigma\tau$ we find that p_n^2 is convergent and hence p_n and k_n are both convergent. Thus $K_{\mathbb{C}}P_+$ is closed in $G_{\mathbb{C}}$. By Lemma 3.10, its Lie algebra is $\mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_+$. □

lma:GKPopen **Lemma 3.12.** *The set $GK_{\mathbb{C}}P_+$ is open in P_-KP_+ and $G \cap K_{\mathbb{C}}P_+ = K$. Similarly, the set $GK_{\mathbb{C}}P_-$ is open in P_+KP_- and $G \cap K_{\mathbb{C}}P_- = K$.*

Proof. **Step 1**. We prove $G \cap K_{\mathbb{C}}P_+ = K$. Suppose that $p \in P = \exp \mathfrak{p}$ has the form

$$p = kp_+ \,, \quad k \in K_{\mathbb{C}}, p_+ \in P_+ \,.$$

Write σ for the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$. Applying the automorphism $\sigma\tau$ we thus get $p^{-1} = kp_+^{-1}$. In particular, $p^2 = p_+^2$. Applying τ again, $p^{-2} \in P_-$. Thus, $p_+^2 \in P_-$. It follows that $p = k = p_+ = 1$. From $G = KP$, we conclude.

Step 2. We prove that $GK_{\mathbb{C}}P_+$ is open in $G_{\mathbb{C}}$. This will conclude the proof as a consequence of Lemma 3.10.

Consider the map $\psi : G \times K_{\mathbb{C}}P_+ \to G_{\mathbb{C}}$ given by

$$\psi(g, x) = gx \,.$$

We need to show that ψ is submersive. Take $Y \in \mathfrak{g}$, $Z \in \mathfrak{k} + \mathfrak{p}_+$. Then

$$\psi(g \exp tY, x) = gx \exp\Big(t \operatorname{Ad}(x^{-1})Y\Big) \,, \quad \psi(g, x \exp tZ) = gx \exp tZ \,.$$

It follows that

$$\mathrm{d}\psi_{(g,x)}(\mathrm{d}L_g Y, \mathrm{d}L_x Z) = \mathrm{d}L_{gx}(\operatorname{Ad}(x^{-1})Y + Z) \,.$$

Here $L_g : G \to G$ is the left translation by g, L_x is similar. The image of $\mathrm{d}\psi$ is thus

$$\mathrm{d}L_{gx} \circ \operatorname{Ad}(x^{-1})(\mathfrak{g} + \mathfrak{k} + \mathfrak{p}_+) = \mathrm{d}L_{gx}\mathfrak{g}_{\mathbb{C}} \,.$$

□

3.4. The embedding theorem.

Theorem 3.13. *There is a holomorphic embedding $G/K \to \mathfrak{p}_+$ onto a bounded symmetric domain in $\mathfrak{p}_+$.*

Proof. Consider the diagram

{eq:embed} (3.2)

$$\begin{tikzcd}
GK_{\mathbb{C}}P_-/K_{\mathbb{C}}P_- \arrow[r] \arrow[d, "\approx"] & P_+K_{\mathbb{C}}P_-/K_{\mathbb{C}}P_- \arrow[r] \arrow[d, "\approx"] & G_{\mathbb{C}}/K_{\mathbb{C}}P_- \\
G/K \arrow[r, dashed] & P_+ \arrow[r, "\log"] & \mathfrak{p}_+
\end{tikzcd} .$$

All maps in question are the natural ones. The first vertical map is a differmorhism by Lemma 3.12, the second by Lemma 3.10. The log map is a diffeomorphism by Proposition 3.9. The map

$$GK_{\mathbb{C}}P_-/K_{\mathbb{C}}P_- \to P_+K_{\mathbb{C}}P_-/K_{\mathbb{C}}P_-$$

is an open immersion by Lemma 3.12. It follows that the map $\psi : G/K \to \mathfrak{p}_+$ is a diffeomorphism onto an open subset. By the computations in [Hel79, Lemm 7.12], the image is a bounded domain.

It remains to check that ψ is holomorphic. It suffices to prove that the map $G/K \to P_+$ is holomorphic. For this purpose, we may assume that D is irreducible. Let $J : \mathfrak{p}_+ \to \mathfrak{p}_+$ be the map

$$J\mathrm{d}\psi(X) = \mathrm{d}\psi(JX) \,, \quad X \in \mathfrak{p} \,.$$

Then by definition J commutes with all $\operatorname{Ad} k$, $k \in K$. But the adjoint representation of K on $\mathfrak{p}_+$ is irreducible. By Schur's lemma, $J = c$ with $c = \pm\mathrm{i}$. Replacing J by $-J$ if necessary, we may assume that $J = \mathrm{i}$. Namely,

{eq:dpsi} (3.3)
$$\mathrm{i}\mathrm{d}\psi(X) = \mathrm{d}\psi(JX) \,, \quad X \in \mathfrak{p} = T_oG/K \,.$$

Observe that the map $G/K \to G_{\mathbb{C}}/K_{\mathbb{C}}P_-$ commutes with the action of G. Observe that $G_{\mathbb{C}}/K_{\mathbb{C}}P_-$ has a natural complex structure, invariant under the $G_{\mathbb{C}}$-action. Moreover, as the map $P_+ \to G_{\mathbb{C}}/K_{\mathbb{C}}P_-$ is holomorphic, it follows that (3.3) also holds for any tangent vector of G/K. So $G/K \to P_+$ is indeed holomorphic. □

Let $U = \exp \mathfrak{u}$.

prop:UoverK **Proposition 3.14.** *The map $U/K \to G_{\mathbb{C}}/K_{\mathbb{C}}P_-$ is a biholomorphic isomorphism.*

Proof. We first prove that $U \cap K_{\mathbb{C}}P_- = K$. Let u be an element in the left-hand side. Then $u^{-1}\sigma\tau(u) \in P_-$. Applying τ, we thus find $\sigma\tau(u) = u$. As U is simply connected, we thus have $u \in K$.

It follows that the map $f : U/K \to G_{\mathbb{C}}/K_{\mathbb{C}}P_-$ is injective. It is also regular and the two sides have the same dimension. It follows that the image is open. From the compactness, we conclude that this map is bijective. Observe that f commutes with U, so in order to show that f is holomorphic, it suffices to prove $\mathrm{d}f_o$ preserves the almost complex structure, which is easy: first observe that for any $X \in \mathfrak{p}$, $\mathrm{d}f_o(\mathrm{i}X) = \mathrm{i}\mathrm{d}\psi(X)$, thus,

$$\mathrm{d}f_0(\mathrm{i}JX) = \mathrm{i}\mathrm{d}\psi(JX) = -\mathrm{d}\psi(X) = \mathrm{i}\mathrm{d}f_0(\mathrm{i}X)\,.$$

Here we identify $T_oU/K \cong \mathrm{i}\mathfrak{p}$. $\square$

In particular, $K_{\mathbb{C}}P_-$ is a parabolic subgroup of $G_{\mathbb{C}}$. Observe that P_- is unipotent, as can be seen from the Lie algebra. Moreover, $K_{\mathbb{C}}$ is reductive as it has a compact real form. It follows that P_- is the unipotent radical of $K_{\mathbb{C}}P_-$ and $K_{\mathbb{C}}$ is a Levi subgroup. Moreover,

$$K_{\mathbb{C}}P_- = K_{\mathbb{C}} \ltimes P_-\,.$$

Theorem 3.15. *There is a natural holomorphic embedding $\mathfrak{p}_+ \to \check{D} = U/K$.*

Proof. The map $P_+K_{\mathbb{C}}P_-/K_{\mathbb{C}}P_- \to G_{\mathbb{C}}/K_{\mathbb{C}}P_-$ in (3.2) is a holomorphic embedding by Lemma 3.10. The theorem therefore follows from Proposition 3.14. $\square$

Let us make the following observation.

op:psiequiv **Proposition 3.16.** *The map $\psi : D \to \mathfrak{p}_+$ is equivariant with respect to the K-action. Here K acts on D by left multiplication and on $\mathfrak{p}_+$ by adjoint action. Similarly, the embedding $\mathfrak{p}_+ \to G_{\mathbb{C}}/K_{\mathbb{C}}P_-$ is also invariant under the adjoint action of K.*

Proof. Take $g \in G$, decomposed as $g = p_+k'p_-$. Then kg can be decomposed as $\mathrm{Ad}(k)(p_+)(kk')p_-$. $\square$

or:dpsiinva **Corollary 3.17.** *The differential $\mathrm{d}\psi : \mathfrak{p} \to \mathfrak{p}_+$ is invariant under the adjoint action of K.*

Now we can make the embedding $D \to \mathfrak{p}_+$ very explicit.

Recall the following elementary lemma:

lma:expsl2c **Lemma 3.18.** *Let (X, Y, H) be a $\mathfrak{sl}_2$-triple in $\mathfrak{sl}(2,\mathbb{C})$. Consider any Lie group L with Lie algebra $\mathfrak{sl}(2,\mathbb{C})$. Then for any $t \in \mathbb{C}$ with $\cosh t \neq 0$,*

$$\exp t(X+Y) = \exp(\tanh t)X \cdot \exp(-\log\cosh t)H \cdot \exp(\tanh t)Y\,. \tag{3.4}$$

Proof. It suffices to prove this for $\mathrm{SL}(2,\mathbb{C})$. In this case, we may identify

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Then we compute[‡]

$$\exp t(X+Y) = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}, \quad \exp(\tanh t)X = \begin{bmatrix} 1 & \tanh t \\ 0 & 1 \end{bmatrix},$$
$$\exp(\tanh t)Y = \begin{bmatrix} 1 & 0 \\ \tanh t & 1 \end{bmatrix}, \quad \exp(-\log\cosh t)H = \begin{bmatrix} (\cosh t)^{-1} & 0 \\ 0 & \cosh t \end{bmatrix}.$$

The desired formula therefore follows. $\square$

In particular,

[‡]To make things easier, recall the following well-known result: given $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, let ω be a square root of $a^2 + bc$. Then $\exp A = \cosh\omega \mathrm{I} + \frac{\sinh\omega}{\omega}A$.

Corollary 3.19. *Let*

$$Z = \sum_{i=1}^{s} t_i x_i \in \mathfrak{a}\,,$$

then

$$\exp Z = \exp X \exp H \exp Y\,,$$

where

$$X = \sum_{i=1}^{s} \tanh t_i X_{\gamma_i}\,, \quad Y = \sum_{i=1}^{s} \tanh t_i X_{-\gamma_i}\,, \quad H = \sum_{i=1}^{s} \log\cosh t_i [X_{-\gamma_i}, X_{\gamma_i}]\,.$$

psiexplicit **Corollary 3.20.** *Under the embedding* $\psi : D \to \mathfrak{p}_+$, *a point* kao *with* $k \in K$, $a = \exp(\sum_i a_i x_i) \in A = \exp \mathfrak{a}$ *is mapped to* $\mathrm{Ad}(k) \sum_i \tanh t_i X_{\gamma_i}$.

Also recall that from Cartan's decomposition, $G = KAK$, so KA acts transitively on D, it follows that this corollary completely determines ψ.

3.5. **Hermann convexity theorem.** In fact, the image of D in $\mathfrak{p}_+$ can be characterized more explicitly.

For any $X \in \mathfrak{p}_+$, define

$$T(X) : \mathfrak{p}_- \to \mathfrak{k}_{\mathbb{C}}\,, \quad Y \mapsto [Y, X]\,.$$

Let $T(X)^* : \mathfrak{k}_{\mathbb{C}} \to \mathfrak{p}_-$ be the adjoint of $T(X)$ with respect to B_τ.

Theorem 3.21 (Hermann convexity theorem)**.** *We have*

$$D = \left\{ X \in \mathfrak{p}_+ : T(X)^* \circ T(X) < 2\mathrm{Id}_{\mathfrak{p}_+} \right\}\,.$$

Here we write X_i for X_{γ_i}.

Write $A = \exp \mathfrak{a}$. By Cartan's KAK decomposition, $KAo = D$. So by Proposition 3.16, as subsets of $\mathfrak{p}_+$,

$$D = \mathrm{Ad}\, K(Ao)\,.$$

We choose the base point corresponding to $1 \in G$.

Lemma 3.22. *We have*

$$Ao = \left\{ \sum_{i=1}^{r} a_i X_i : a_i \in (-1, 1) \right\}\,.$$

Proof. In fact, for $t_i \in \mathbb{R}$,

$$\exp\left(\sum_i t_i x_i\right) = \exp\left(\sum_i \tanh t_i X_{\gamma_i}\right) \exp\left(-\sum_i \log(\cosh t_i)[X_{\gamma_i}, X_{-\gamma_i}]\right) \exp \sum_i \tanh t_i X_{-\gamma_i}\,.$$

Hel79
See [Hel79, Lemma VIII.7.11] for example. The desired result follows. □

It follows that in order to prove the theorem, it suffices to consider $X = \sum_i a_i X_i$ with $a_i \in \mathbb{R}$ and show that

$$T^*(X) \circ T(X) < 2\mathrm{Id} \Leftrightarrow |a_i| < 1$$

or equivalently

$$\|T(X)\| < \sqrt{2} \Leftrightarrow |a_i| < 1\,.$$

For this purpose, we need an explicit computation. Take $Y \in \mathfrak{p}_-$. From the root decomposition to be considered in the next note, we know that Y admits a decomposition

{eq:Y}
$$Y = \sum_i b_i X_{-i} + \sum_i \sum_{\alpha \in P_i} b_\alpha X_{-\alpha} + \sum_{i<j} \sum_{\alpha \in P_{ij}} c_\alpha X_{-\alpha}\,. \tag{3.5}$$

Here P_i (P_{ij}) is the set of positive non-compact roots compatible with $\frac{1}{2}\gamma_i$ ($\frac{1}{2}(\gamma_i + \gamma_j)$) (in the sense that the restriction to $\mathfrak{a}' = \sum \mathbb{R} h_i$ are equal). Then

{eq:commXY}
$$[X, Y] = \sum_i a_i b_i [X_i, X_{-i}] + \sum_i \sum_{\alpha \in P_i} a_i b_\alpha [X_i, X_{-\alpha}] + \sum_{i \neq j} \sum_{\alpha \in P_{ij}} a_i c_\alpha [X_i, X_{-\alpha}]\,. \tag{3.6}$$

From our normalization of X_α, $\tau(X_\alpha)=-X_{-\alpha}$. It follows that for $\alpha\neq\beta$, $B_\tau(X_\alpha,X_\beta)=0$. By Jacobi identity,

$$B_\tau([X_i,X_{-\alpha}],[X_i,X_{-\beta}])=\alpha(h_i)B(X_{-\alpha},X_\beta)\,.$$

In particular, the terms in (3.5), (3.6) are orthogonal to each other. So the square norms of Y and $[X,Y]$ are just square sums of the coefficients. From this, the assertion follows.

3.6. **Embedding into the compact dual.** The final topic concerns the embedding $\mathfrak{p}_+\to\check{D}$.

Theorem 3.23. *The embedding $\mathfrak{p}_+\to\check{D}$ realizes $\mathfrak{p}_+$ as a Zariski open dense subset of $\check{D}$.*

References

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