NOTES ON SHIMURA VARIETIES X. PEIRCE DECOMPOSITION

MINGCHEN XIA

Contents

1. Introduction

This is one of a series of notes prepared for a seminar on the toroidal compactifications of Shimura varieties.

2. Some complements about the last time

ec:lasttime

We need to recall and refine a few results from Yuanyang's lecture.

2.1. The automorphism group of cones. Let V be a real vector space of finite dimension and C be a symmetric cone. In the old terminology of [AMRT AMSRT10], they say C is an (open) self-adjoint homogeneous cone.

We know from last time that $G := \mathrm{Aut}(V, C) \subsetneq \mathrm{Aut}(V)$ is a closed subgroup in the Euclidean topology and is real reductive. It is not clear to me if this implies that $\mathrm{Aut}(V, C)$ is algebraic. *

We assume that there is a Zariski closed subgroup $\mathcal{G}_{\mathbb{R}}$ of $\mathrm{GL}(V)$ such that $\mathrm{Aut}(V, C)^+ = \mathcal{G}_{\mathbb{R}}(\mathbb{R})^+$. Unfortunately, I have no idea how to prove this, nor can I find out a proof in the literature!

Also observe that $\mathrm{Aut}(V, C)$ is not connected in general. For example, when C is the product of two simple cones. One has to be extra careful when reading [AMRT AMSRT10]!

From the last time, we know that (V, C) together with a fixed $p \in C$ corresponds to a Euclidean† Jordan algebra on V with unit p. Under this correspondence, $\bar{C} = \{x^2 : x \in V\}$ and $C = \{x^2 : x \in V \text{ is invertible}\}$.

When the Jordan algebra V is defined over $\mathbb{Q}$ (in the sense that we choose a specific Jordan algebra V' over $\mathbb{Q}$ so that $V'_{\mathbb{R}} \cong V$ as (unital) Jordan algebras), we say (V, C) is defined over $\mathbb{Q}$. In this case, we assume that $\mathcal{G}_{\mathbb{R}}$ has a canonical $\mathbb{Q}$-structure $\mathcal{G}$.

One may think that the existence of $\mathcal{G}$ follows from a simple Galois descent, but it is not clear at all to me why $\mathrm{Gal}(\mathbb{R}/\mathbb{Q})$ acts on $\mathcal{G}_{\mathbb{R}}$. Equally unclear to me is what functor $\mathcal{G}$ represents.

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*On Mathoverflow, the user "Echo" claims that this holds automatically with a proof which I do not understand.

†Formally real in the terminology of [AMRT AMSRT10].

2.2. **Structure algebra.** Fix a point $p \in C$, let K be the isotropy subgroup of p in G. Then K is a maximal compact subgroup. We have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

and $\mathfrak{p}$ is canonically identified with V.

Also recall that in terms of the Jordan algebra V, $\mathfrak{g}$ is the *structure algebra* $\operatorname{str} V$ of V. Recall that

$$\operatorname{str} V = \operatorname{Der} V \oplus V\,.$$

Moreover, $\operatorname{Der} V \cong \mathfrak{k}$. Also we know that all derivations are inner: $\operatorname{Der} V = [V, V]$.

The goal of today is to prove that the flags of boundary components of C is the same as parabolic subgroups of $\mathcal{G}$. The situation is very similar to the case of bounded symmetric domains that we have already handled.

We will focus on the parts where [AMSRT10] contains mistakes or lacks details.

3. Peirce decomposition

3.1. **Classical Peirce decomposition.** We recall that in a commutative ring R, given any idempotent $e \in R$, we have

$$R = e^2 R \oplus e(1-e)R \oplus (1-e)^2 R.$$

This is the traditional *Peirce decomposition*. We will extend this construction to the non-associative Jordan algebras following Albert (Annals, 1947).

3.2. **Peirce decomposition of Jordan algebras.** Let V be a Jordan algebra defined over $\mathbb{R}$ with unit p, recall that by definition, V is a finite-dimensional non-associative commutative $\mathbb{R}$-algebra such that

(1) p is a unit in V.
(2) $a^2(ba) = (a^2 b)a$ for all $a, b \in V$. (Jordan identity)

For $x \in V$, recall $L_x : V \to V$ is defined as $y \mapsto xy$.

Fix an idempotent $\epsilon \in A$.

Proposition 3.1. *L_ϵ is semi-simple with eigenvalues $0, 1/2, 1$. In fact, $\varphi(L_\epsilon) = 0$, where $\varphi(T) = 2T^3 - 3T^2 + T$.*

Proof. A straightforward computation using the identity $S = T$ from the last lecture shows

$$L_{x^{k+2}} = 2L_{x^{k+1}} L_x + L_{x^2} L_{x^k} - L_{x^k} L_x^2 - L_x^2 L_{x^k}\,.$$

In particular, if $\epsilon \in A$ is idempotent, taking $k = 1$, we find

$$L_\epsilon = 2L_\epsilon^2 + L_\epsilon^2 - L_\epsilon^3 - L_\epsilon^3\,.$$

Namely, $\varphi(L_\epsilon) = 0$. □

In general, a semi-simple operator is diagonalizable only after passing to the algebraic closure, but here we are in a much simpler situation.

Corollary 3.2. *The space V can be decomposed into the eigenspaces of L_ϵ:*

$$V = V_0 \oplus V_{1/2} \oplus V_1, \quad V_i = \varphi_i(L_\epsilon)V\,. \tag{3.1}$$

Observe that when V and p are both defined over $\mathbb{Q}$, so is this decomposition.

(3.1) is the *Peirce decomposition* of V with respect to ϵ. When there is a risk of confusion, we also write $V_i(\epsilon)$ instead of V_i. Observe that $V_1(\epsilon) = V_0(p - \epsilon)$.

Proof. The three roots of φ are $0, 1/2, 1$. We decompose

$$\varphi(T) = T\varphi_0(T) = -\frac{1}{2}(T - 1/2)\varphi_{1/2}(T) = (T-1)\varphi_1(T)\,.$$

The unusual normalization of $\varphi_1(T)$ guarantees that

$$\varphi_0 + \varphi_1 + \varphi_2 = 1\,.$$

In particular, any $x \in V$ can be decomposed as $x_0 + x_{1/2} + x_1$ according to the ϵ-eigenvalue and $x_i = \varphi_i(L_\epsilon)x$. $\square$

We recall the following basic fact:

Lemma 3.3. *Let $x, y \in A$, then x and y^2 strictly commute iff xy and y strictly commute.*

Corollary 3.4. *L_x for $x \in V_0$ or $x \in V_1$ preserves each V_i.*

Proof. By Lemma 3.3, when $x \in V_0$, $[L_x, L_\epsilon] = 0$ and the claim is proved in this case.
Similarly, when $x \in V_1$, $p - \epsilon$ is an idempotent, so $[L_x, L_{p-\epsilon}] = 0$. $\square$

In particular, $V_0 \cdot V_1 = 0$: take $a \in V_0$, $b \in V_1$, then $L_\epsilon(ab) = L_a L_\epsilon(b) = L_a(b) = ab$. Similarly, $L_\epsilon(ba) = 0$. So $ab = 0$.

A similar manipulation shows that the multiplication in V has the following form:

Proposition 3.5. *The multiplication table of V is*

(1) $V_0V_0 \subseteq V_0$, $V_1V_1 \subseteq V_1$, $V_0V_1 = 0$.
(2) $V_0V_{1/2} \subseteq V_{1/2}$, $V_1V_{1/2} \subseteq V_{1/2}$.
(3) $V_{1/2}V_{1/2} \subseteq V_0 + V_1$.

We have proved (1) and (2), the proof of (3) is slightly more complicated, see [AMSRT10, Page 46].

In particular, with respect to the trace form: $(x, y) = \mathrm{Tr}(L_{xy})$, the Peirce decomposition (3.1) is orthogonal.

The Peirce decomposition can also be carried out for a commuting family of idempotents in V.

Let $\mathcal{G}$, K, $\mathfrak{p}$ be as in the previous section,

Proposition 3.6. *Assume that V is defined over $\mathbb{Q}$. Let $\epsilon_1, \ldots, \epsilon_n \in V$ be a family or orthogonal ($\epsilon_i\epsilon_j = \delta_{ij}\epsilon_i$) idempotents defined over $\mathbb{Q}$, then $\sum_i \mathbb{R}\epsilon_i$ is the Lie algebra of a unique $\mathbb{Q}$-split torus of rank k in $\mathcal{G}$ contained in $\exp \mathfrak{p}$.*

Conversely, any maximal $\mathbb{Q}$-split tori of $\mathcal{G}$ contained in $\exp \mathfrak{p}$ arise in this way.

This result is proved in [AMSRT10].

4. Boundary components

Let V be a Euclidean Jordan algebra defined over $\mathbb{Q}$ with unit p. We fix an idempotent $\epsilon \in V$ and form the Peirce decomposition (3.1). Now the V_i's are $\mathbb{Q}$-vector spaces, in contrast to the last section. Let C be the corresponding symmetric cone in $V_\mathbb{R}$.

Observe that V_0 and V_1 are both Euclidean Jordan algebras with $p - \epsilon$ and ϵ as the identities respectively.

Let C_i $(i = 0, 1)$ be the cone of squares of invertible elements in $V_{i,\mathbb{R}}$. They admit canonical $\mathbb{Q}$-model by descent, which we denote by C_i. From the last lecture, we know that C_0 and C_1 are both self-adjoint homogeneous cones. Moreover, using a simple topological argument, the closures of these cones consist of squares of all elements (not necessarily invertible).

We call C_i a *rational boundary component* of C.

Intuitively, ϵ is an idempotent linear operator on the vector space V preserving $\bar{C}$. It is a projection on $V_0 \oplus V_1$ and a rescaling on its orthogonal complement. The latter is not relevant for the boundary behaviour. Then V_0 corresponds to the kernel of the projection and V_1 is the fixed point set. It is then clear that C_0 and C_1 are really boundary components of C in the intuitive sense.

From this intuitive argument, we clearly have

Proposition 4.1. *If $\pi_0 : V \to V_0$ denotes the projection with respect to the Peirce decomposition (3.1). Then $\pi_0(C) = C_0$.*

Proof. We refer to [AMSRT10, Lemma 3.3] for the rigorous proof. $\square$

Proposition 4.2. *The closure $\overline{C}$ is the disjoint union of rational boundary components.*

Proof. Take $y \in \bar{C}$ and $y = x^2$ for some $x \in V$. Let W be the the subalgebra of $V_{\mathbb{R}}$ consisting of polynomials in x without constant term. Then $W \subseteq \mathbb{R}[x]$. As $\mathbb{R}[x]$ is Euclidean (as a Jordan algebra), it is isomorphic to a product of $\mathbb{R}$ with componentwise multiplication, as explained in the last lecture. But then, a non-unital $\mathbb{R}$-subring of $\mathbb{R}[x]$ is necessarily isomorphic to a product copies of $\mathbb{R}$ and *a fortiori* is necessarily unital. In particular, W has a unit e. Now it is clear that $W \subseteq V_1(e)_{\mathbb{R}}$.

Expand $e = a_1 x + a_2 x^2 + \cdots + a_n x^n$ for some $a_i \in \mathbb{R}$. Taking square, we have

$$x(a_1^2 x + \cdots) = e\,.$$

In particular, x is invertible in $V_1(e)_{\mathbb{R}}$. Therefore x lies in a rational boundary component.

It remains to see that boundary components do not overlap. Suppose y is an invertible element in $V_1(\epsilon_1)_{\mathbb{R}}$ and $V_1(\epsilon_2)_{\mathbb{R}}$.

Take $x_i \in V(\epsilon_i)_{\mathbb{R}}$ such that $x_i y = \epsilon_i$. From the last lecture, the inverse of y is necessarily in $W = \mathbb{R}[y] \subseteq V_{\mathbb{R}}$.[‡] Applying this to $V_1(\epsilon_i)$, we find that $x_i \in W$ and hence $\epsilon_i \in W$, It follows that $\epsilon_1 = \epsilon_1\epsilon_2 = \epsilon_2$. □

5. Parabolic subgroups and boundary components

We use the same notations as in the last section.

5.1. Parabolic subgroups and normalizers. Fix an idempotent $\epsilon \in V$. We could regard ϵ as a cohcaracter $a \in X_*(\mathcal{G})_{\mathbb{Q}}$ induced by $\exp(t) \mapsto L_{\exp(t\epsilon)}$.

We have reviewed the dynamical theory of parabolic subgroups in the previous lectures. Let $\mathcal{P}(a)$ be the parabolic subgroup corresponding to a:

$$\mathcal{P}(a)(R) = \{x \in \mathcal{G}(R) : \lim_{s\to 0} a(s)^{-1} x a(s) \text{ exists}\}.$$

Write $\mathcal{Z}(a)$ and $\mathcal{U}(a)$ for the center and unipotent radical of $\mathcal{P}(a)$. As usual, the cocharacter allows us to decompose

$$\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n\,,$$

where on $\mathfrak{g}_n$, the conjugate action of $a(s)$ acts as s^n.

Next consider a rational boundary component C_0 of C. Let L be the $\mathbb{R}$-linear span of C_0. Let $\mathrm{Aut}(V, L)$ be the group of automorphisms of V preserving L. The group $\mathrm{Aut}(V, L)$ is clearly representable and we have an algebraic surjective morphism $\mathrm{Aut}(V, L) \to \mathrm{Aut}(L)$ defined over $\mathbb{Q}$. From Section 2, we know that the automorphism group $\mathcal{G}'$ of (L, C_0) is a closed algebraic subgroup of $\mathrm{Aut}(L)$, so the fiber product functor $\mathcal{N}(C_0) := \mathrm{Aut}(V, L) \times_{\mathrm{Aut}(L)} \mathcal{G}'$ is representable. We call $\mathcal{N}(C_0)$ the *normalizer* of C_0. [§]

prop:PNC0 **Proposition 5.1.** *We have*

$$\mathcal{P}(a) = \mathcal{N}(C_0)\,.$$

Proof. We first show $\mathcal{P}(a) \subseteq \mathcal{N}(C_0)$. As $\mathcal{P}(a)$ is parabolic, it is connected. So by [Mil17, Theorem 17.93], $\mathcal{P}(a)(\mathbb{Q})$ is dense in $\mathcal{P}(a)$.

It suffices to prove the inclusion on $\mathbb{Q}$-points.

First assume that $u \in \mathfrak{g}_m$ for some $m > 0$, then

$$s^m a(s) u = u a(s)\,.$$

Take the limit $s \to 0$, we have

$$u\pi_0 = 0\,.$$

It follows that $uC_0 = u\pi_0(C) = 0$. In particular, $\mathcal{U}(a)$ normalizes C_0. On the other hand, if $z \in \mathcal{Z}(a)(\mathbb{Q})$, then z commutes with $a(s)$ for all s, hence z commutes with π_0 as well. So z also normalizes C_0. Therefore $\mathcal{P}(a) \subseteq \mathcal{N}(C_0)$.

Now we know that $\mathcal{N}(C_0)$ is also parabolic, so it is connected as well. Again by [Mil17, Theorem 17.93], for the reverse direction, it suffices to consider the $\mathbb{Q}$-points.

[‡]The definition of W is not correct in [AMSRT10].

[§]The representability and the correct definition of $\mathcal{N}(C_0)$ are both omitted in [AMSRT10].

Assume that the two groups are different. Then there is some $m < 0$ so that $\mathfrak{g}_m \cap \mathfrak{n}(C_0) \neq 0$. Pick up $\nu \neq 0$ in the intersection. Then $a(s)\nu = s^{-m}\nu a(s)$. Let $s \to 0$, we have $\pi_0 \nu = 0$. So $\pi_0 \circ \exp \nu = \pi_0$. By definition, $\exp \nu \in \mathcal{N}(C_0)(\mathbb{R})$. In particular, $\exp \nu(p - \epsilon) = \pi_0(p - \epsilon) = p - \epsilon$.

The same argument applies to $p - \epsilon$ shows that $(\exp \nu)(\epsilon) = \epsilon$.

Adding up, we have $(\exp \nu)p = p$ and $\exp \nu \in K$. This contradicts the fact that K is compact and $\mathfrak{g}_m$ is nilpotent. □

Corollary 5.2. *The action of $\mathcal{G}(\mathbb{Q})_0$ permutes the rational boundary components of V.* ¶

Here $\mathcal{G}(\mathbb{Q})_0 = \mathcal{G}(\mathbb{Q}) \cap \mathcal{G}(\mathbb{R})^+$.

Proof. By Cartan decomposition and Proposition 5.1, $\mathcal{G}(\mathbb{R})^+ = K\mathcal{N}(C_0)(\mathbb{R})$. In fact, when $\mathcal{G}$ is a closed subgroup of $\mathrm{GL}(V_\mathbb{R})$, the Cartan decomposition can be induced from the standard Cartan decomposition of $\mathrm{GL}(V_\mathbb{R})$. So we may assume that $g \in \mathcal{G}(\mathbb{Q})_0$ can be represented as kp with $k \in K$ and $p \in \mathcal{N}(C_0)(\mathbb{R})$, both being matrices with rational coefficients. Only k is relevant to this proposition, so let us assume that in addition g fixes p. Then $g : V \to V$ is an automorphism of Jordan algebras by a direct computation, so $g\epsilon$ is also an idempotent and $gC_0(\epsilon) = C_0(g\epsilon)$. □

As in the case of bounded symmetric domains, we decompose C into irreducible pieces.

Lemma 5.3. *The cone C admits a unique (up to permutation) decomposition into $C_1 \oplus \cdots \oplus C_n$ for $\mathbb{Q}$-simple cones, where C_i are rational symmetric cones in some subspace $V_i \subseteq V$ satisfying $V = V_1 \oplus \cdots \oplus V_n$.*

Here $\mathbb{Q}$-simple means C cannot be further decomposed. The rationality of a cone refers to the rationality of the corresponding Jordan algebra (with respect to any choice of identity.)

Proof. The existence of such a decomposition is trivial. If $C = C_1 + C_2 = D_1 + D_2$ are two decompositions whose components are not necessarily $\mathbb{Q}$-simple, we claim that

$$C_1 = (C_1 \cap D_1) \oplus (C_1 \cap D_2)\,.$$

From this, we easily conclude the lemma. Let $x \in C_1$, then $x = d_1 + d_2$ with $d_i \in D_i$. Write $d_i = e_{i1} + e_{i2}$ with $e_{ij} \in C_j$. Then $x - e_{11} - e_{21} = e_{12} + e_{22}$ is in the linear span of C_1 and of C_2, so $e_{12} + e_{22} = 0$. It follows that $e_{12} = e_{22} = 0$ and $d_1, d_2 \in C_1$. □

5.2. **Flags of boundary components.** Next we assume that C is $\mathbb{Q}$-simple. We will establish the correspondence between parabolic subgroups and boundary components.

Fix a maximal set $\epsilon_1, \ldots, \epsilon_n$ of orthogonal idempotents in V. Let $a_j(s)$ denote the cocharacter defined by $f_j = \epsilon_1 + \cdots + \epsilon_j$. Set $C_j = C_0(f_j) = a_j(0)C$.

Observe that $V_0(f_{j+1})$ is a proper subspace of $V_0(f_j)$. In fact, if $x \in V_0(f_{j+1})$, then $0 = f_{j+1}x = f_j x + \epsilon_{j+1}x$. But $\epsilon_{j+1} \in V_1(f_{j+1})$ so $\epsilon_{j+1}x = 0$ (c.f. the multiplication table). Thus $f_j x = 0$ and $x \in V_0(f_j)$. The containment is strict as $\epsilon_{j+1} \in V_0(f_j) \setminus V_0(f_{j+1})$.

It follows that $\bar{C}_{j+1}$ is a proper subset of $\bar{C}_j$. We call

$$0 = \bar{C}_n \subset \bar{C}_{n-1} \subset \cdots \subset \bar{C}_0 = \bar{C}$$

the *standard flag* of boundary components. An element in the standard flag is a *standard boundary component*.

Let $\mathcal{A}$ be the maximal $\mathbb{Q}$-split torus defined by $\epsilon_1, \ldots, \epsilon_n$.

Proposition 5.4. *For any cocharacter $b \in X_*(\mathcal{G})_\mathbb{Q}$ such that $b(0) := \lim_{s\to 0} b(s)$ exists in $\mathcal{E}\mathrm{nd}(V)$ and is non-zero, $b(0)C$ is the image of a standard boundary component under $g \in \mathcal{G}(\mathbb{Q})_0$.*

Proof. ‖ Up to conjugation by some element in $\mathcal{G}(\mathbb{Q})_0$, we may assume that $b \in X_*(\mathcal{A})_\mathbb{Q}$. From the root structure to be studied later in Proposition 6.1, upon replacing b by a Weyl conjugate, we may assume that $b = \prod_i a_i^{m_i}$ with m_i decreasing (this condition defines a Weyl chamber). As $b(0)$ exists and $b(s)\epsilon_n = s^{2m_n}\epsilon_n$, we have $m_n \geq 0$. Let m_j be the last non-zero element in m_i. Then $b(0)C = C_j$. □

¶Again, this corollary is not correctly stated in [AMSRT10].

‖The original proof in [AMSRT10] identifies a wrong Weyl chamber.

It follows that the standard flag is a maximal flag of rational boundary components by dimension comparison.

A flag of boundary components is a sequence $C_s, \dots, C_0 = C$ of boundary components such that

$$\bar{C}_s \subseteq \cdots \subseteq \bar{C}_0 \,.$$

By a similar proof, we have

Proposition 5.5. *Any flag of rational boundary components is the image under $\mathcal{G}(\mathbb{Q})_0$ of some subflag of the standard flag.*

Here we require that two adjcent elements in the flag are different.

Proof. Let

$$\bar{C}_s \subset \bar{C}_{s-1} \subset \cdots \subset \bar{C}_1 \subset \bar{C}$$

be the flag. Let (V_i, d_i) be the Jordan algebra associated with C_i. Then

$$E = \{p - d_1, d_1 - d_2, \dots, d_{s-1} - d_s, d_s\}$$

is a set of orthogonal idempotents and they generate a sub-Jordan algebra. The corresponding torus $\mathcal{B}$ is $\mathbb{Q}$-split, so up to conjugation, we may assume that $\mathcal{B} \subseteq \mathcal{A}$.

From $\mathfrak{b} \subseteq \mathfrak{a}$, up to Weyl action, we may assume that there is an increasing map $\varphi : \{1, \dots, s\} \to \{1, \dots, n-1\}$ so that $d_i = f_{\varphi(i)}$. The desired result follows. $\square$

Theorem 5.6. *There is a bijection between the set of flags of rational boundary components and the set of parabolic subgroups of $\mathcal{G}$.*

The forward direction sends

$$\bar{C}_s \subset \bar{C}_{s-1} \subset \cdots \subset \bar{C}_1 \subset \bar{C}$$

to $\bigcap_i \mathcal{N}(C_i)$. We denote this map by Φ.

Proof. Let $\mathcal{P}$ be the minimal $\mathbb{Q}$-parabolic subgroup of $\mathcal{G}$ corresponding to the set of simple roots $\{(\epsilon_i^* - \epsilon_{i+1}^*)/2\}$. As it is contained in the maximal parabolic subgroups defined by the standard flag, we have

$$\mathcal{P} = \bigcap_i \mathcal{N}(C_i) \,,$$

where C_i denotes the standard flag. In particular, by conjugation, Φ is well-defined.

It follows from the standard structure theory that Φ is surjective: it suffices to consider a standard parabolic subgroup. It is well-known that a standard parabolic subgroup is the intersection of maximal standard parabolic subgroups containing it. So surjectivity is clear.

As for the injectivity, as any parabolic subgroup can be uniquely decomposed as the intersection of maximal parabolic subgroups, it suffices to show that for any two boundary components C_a and C_b if $\mathcal{N}(C_a) = \mathcal{N}(C_b)$, then $C_a = C_b$. In fact, assume that C_a (resp. C_b) is conjugate to the standard boundary component C'_a (resp. C'_b) by g^{-1} (resp. g'^{-1}). Then $\mathcal{N}(C'_a)$ is conjugate to $\mathcal{N}(C'_b)$ by our assumptions. But standard maximal parabolic subgroups do not conjugate with each other, so $C'_a = C'_b$. Therefore

$$g'^{-1} g \mathcal{N}(C_i) (g'^{-1} g)^{-1} = \mathcal{N}(C_i) \,.$$

But as parabolic subgroups are self-normalizing, $g'^{-1} g \in \mathcal{N}(C_i)(\mathbb{Q})$. So

$$C_b = g' C'_b = g C'_a = C_a \,.$$

$\square$

6. Root computations

We continue to assume that V is defined over $\mathbb{Q}$.

Choose a maximal set $\epsilon_1, \dots, \epsilon_n$ of orthogonal idempotents in V. By Proposition 3.6, they define a maximal $\mathbb{Q}$-split torus $\mathcal{A}$ in $\mathcal{G}$. The goal of this section is to compute the root structure of $(\mathcal{G}, \mathcal{A})$.

First observe that $\epsilon_1 + \cdots + \epsilon_n = 1$ as otherwise, $1 - \epsilon_1 - \cdots - \epsilon_n$ can be added to the list. In particular, $L_{\epsilon_1} + \cdots + L_{\epsilon_n} = \mathrm{id}$. On the other hand, by Lemma 3.3, $[L_{\epsilon_i}, L_{\epsilon_j}] = 0$. So we have a commuting family of semi-simple operators. This allows us to decompose V into

$$V = \bigoplus_{r \leq s} V_{rs}, \quad \epsilon_t \text{ acts as } (\delta_{tr} + \delta_{ts})/2 \text{ on } V_{rs}\,.$$

We will set $V_{rs} = V_{sr}$. We identify $\mathfrak{g}$ with the structure algebra $\operatorname{str} V$.

prop:root **Proposition 6.1.** *The root space decomposition of* $(\mathfrak{g}, \mathfrak{a})$ *is given by*

$$\mathfrak{g} = Z(\mathfrak{a}) \oplus \bigoplus_{i \neq j} \mathfrak{g}_{ij}\,,$$

where

$$\mathfrak{g}_{ij} = \{(D, x) \in \operatorname{str} V : x \in V_{ij}, D = -2[L_x, L_{\epsilon_i}]\}\,.$$

Moreover,

$$Z(\mathfrak{a}) = \{D \in \operatorname{Der} V : D\epsilon_i = 0 \text{ for all } i\} \oplus \bigoplus_{i=1}^{n} V_{ii}\,.$$

Moreover, if V is $\mathbb{Q}$-simple, the $\mathbb{Q}$-roots are

$$\left\{ \frac{1}{2}(\epsilon_i^* - \epsilon_j^*) \text{ for } i \neq j \right\}\,.$$

The Weyl group is the permutation group of the ϵ_i's.

So the root system of $(\mathfrak{g}, \mathfrak{a})$ is A_n.

Proof. The proof is a lengthy but straightforward computation, which we omit. □

References

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Mil17 [Mil17] J. S. Milne. Algebraic groups: the theory of group schemes of finite type over a field. Vol. 170. Cambridge University Press, 2017.

Mingchen Xia, Department of Mathematics, Chalmers Tekniska Högskola, Göteborg
Email address, xiam@chalmers.se
Homepage, http://www.math.chalmers.se/~xiam/.