NOTES ON SHIMURA VARIETIES XIII. SIEGEL DOMAINS

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Contents

1. Introduction

This is one of a series of notes prepared for a seminar on the toroidal compactifications of Shimura varieties.

sec:toy

2. Compactification problem for Siegel domain of the first kind

Let $N_{\mathbb{R}}$ be a finite dimensional $\mathbb{R}$-linear space. We do not choose any rational structure on $N_{\mathbb{R}}$.

Let $C \subseteq N_{\mathbb{R}}$ be a symmetric cone with a fixed base point e. Set

$$\overline{G} := \mathrm{Aut}(C, N_{\mathbb{R}})^+ ,$$

which is a real Lie group. Write $K \subseteq \overline{G}$ as the isotropy subgroup of e. Let σ be the Cartan involution of G fixing K.

2.1. Tube domains and bounded symmetric domains.

We now explain the relation between the two objects we have been studying in the whole seminar: symmetric cones and bounded symmetric domains.

Definition 2.1. The open set

$$U := N_{\mathbb{R}} + \mathrm{i}C \subseteq N_{\mathbb{C}}$$

is called the *tube domain* associated with C.

Remark 2.2. More generally, for any open set $C \subseteq N_{\mathbb{R}}$, the domain U is known as a tube domain.

When C is a non-empty open convex cone $C \subseteq \mathbb{R}^m$ not containing a full line, U is known as a *Siegel domain of the first kind.*

Proposition 2.3. *U is a bounded symmetric domain.*

Proof. **Step 1**. U is a homogeneous domain.

Clearly $\overline{G} \ltimes N_{\mathbb{R}}$ acts transitively on U. Here we take the canonical inclusion $\overline{G} \subseteq \mathrm{GL}(N_{\mathbb{C}})$.

Step 2. There is an involutive symmetry ι of U around $\mathrm{i}e$.

Recall that there is a canonical real Jordan algebra structure on $N_{\mathbb{R}}$ with e as identity. By scalar extension, we get a complex Jordan algebra structure on $N_{\mathbb{C}}$ with identity e.

The required symmetry is given by $\iota(x) = -x^{-1}$. It is not hard to see that ι is a local involutive symmetry near $\mathrm{i}e$. It remains to show that if $x \in U$, then so is $-x^{-1}$.

Step 2.1. First observe that $\overline{G}$ preserves invertible elements. To see this, recall that we have seen previously that if $g \in \overline{G}$, then

$$(ge)^{-1} = \sigma(g)e\,.$$

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In particular, if $g, g_1 \in \overline{G}$, then

$$(gg_1(\mathrm{i}e))^{-1} = -\mathrm{i}\sigma(gg_1)e = -\mathrm{i}\sigma(g)\sigma(g_1)e = \sigma(g)(g_1(\mathrm{i}e))^{-1}\,.$$

As $g_1(\mathrm{i}e)$ runs over $\mathrm{i}C$, by analytic continuation, we have

$$(gx)^{-1} = \sigma(g)x^{-1}$$

for any invertible element $x \in U$. We conclude our observation.

So the problem is reduced to show that $-(x+\mathrm{i}e)^{-1} \in U$ for any $x \in N_{\mathbb{R}}$.

Step 2.2. We reduce to the case $N_{\mathbb{R}} = \mathbb{R}^n$ and $C = \mathbb{R}^n_{>0}$. So that $U = \mathbb{H}^n$.

Let $\mathbb{R}[x] \subseteq N_{\mathbb{R}}$ be the real Jordan subalgebra generated by x and e. From the fact that $\mathbb{R}[x]$ is formally real, we know that

$$\mathbb{R}[x] \cong \bigoplus_{i=1}^{n} \mathbb{R}\epsilon_i$$

for some orthogonal idempotents ϵ_i. It is straightforward to verify that

$$\mathbb{R}[x] \cap C \cong \sum_{i=1}^{n} \mathbb{R}_{>0}\epsilon_i$$

under the isomorphism as above.

We can clearly replace $N_{\mathbb{R}}$ with $\bigoplus_{i=1}^{n} \mathbb{R}\epsilon_i$ and C with $\sum_{i=1}^{n} \mathbb{R}_{>0}\epsilon_i$.

Step 2.3 It remains to show that for any $x \in \mathbb{R}^n$,

$$-(a+\mathrm{i})^{-1} \in \mathbb{H}^n\,.$$

As the Jordan algebra structure on $\mathbb{R}^n$ is induced by the direct sum of Jordan algebra structures on the $\mathbb{R}$-factors, we can assume $n = 1$. Then in this case, the result is trivial.

Now we have shown that U is a Hermitian symmetric space.

Step 3. To see U is bounded, it suffices to observe that we can choose a linear coordinate so that $C \subseteq \mathbb{R}^n_{>0}$, then $U \subseteq \mathbb{H}^n$. □

2.2. **Quotients.** We make further assumptions:

(1) Assume that there is a $\mathbb{Q}$-algebraic group $\mathcal{G}$ such that

$$G := \mathcal{G}(\mathbb{R})^+ = \mathrm{Aut}(U)^+\,.$$

(2) There is a $\mathbb{Q}$-algebraic subgroup $\mathcal{P} \subseteq \mathcal{G}$ such that

$$P := \mathcal{P}(\mathbb{R})^+ = \overline{G} \ltimes N_{\mathbb{R}}\,.$$

Fix an arithmetic subgroup $\Gamma \subseteq \mathcal{G}(\mathbb{Q}) \cap G$. Set

$$N_{\mathbb{Z}} := \{n \in N_{\mathbb{R}} : n \in \Gamma\}\,.$$

Then $N_{\mathbb{Z}}$ is a lattice in $N_{\mathbb{R}}$. Let $\overline{\Gamma} := (\Gamma \cap P)/N_{\mathbb{Z}}$ be the image of Γ via $P \to \overline{G}$.

Keep in mind: groups related to C are denoted by an extra overline from their lifts to groups related to U.

The goal is to study the arithmetic quotient U/Γ and its compactification. We recall that we have the torus $T = N_{\mathbb{C}}/N_{\mathbb{Z}}$ and an exact sequence

$$0 \to T_c \to T \xrightarrow{\mathrm{ord}} N_{\mathbb{R}} \to 0\,.$$

Here the map ord is defined by taking the imaginary part and $T_c = N_{\mathbb{R}}/N_{\mathbb{Z}}$. We set $U' = \mathrm{ord}^{-1}(C)$. More generally, for any $\epsilon \in C$, we set $U'_\epsilon := \mathrm{ord}^{-1}(C_\epsilon)$ with $C_\epsilon = C + \epsilon$.

We fix an extra data of a $\overline{\Gamma}$-admissible polyhedral decomposition $\Sigma = \{\sigma_\alpha\}_\alpha$ of C. We begin our study of the compatification problems at the cusps.

Let X_Σ be the toric variety defined by Σ. It carries a natural $\overline{\Gamma}$-action as $\overline{\Gamma}$ acts on Σ. If we set U'' as the interior of the closure of U' in X_Σ, then U'' is preserved by the action of $\overline{\Gamma}$.

We set U''_ϵ as the interior of the closure of U'_ϵ in X_Σ.

Theorem 2.4. *The group $\overline{\Gamma}$ acts properly discontinuously on U'' and the image of U''_ϵ in $U''/\overline{\Gamma}$ is open and relatively compact.*

The compactification of U/Γ at the cusp $i\infty$ is done by gluing $U\Gamma$ with $\overline{\Gamma}U''_\epsilon/\overline{\Gamma}$ * along $\overline{\Gamma}U'_\epsilon/\overline{\Gamma}$. A key point is to show that $\overline{\Gamma}U'_\epsilon/\overline{\Gamma} \subseteq U/\Gamma$, namely, for a large enough $\epsilon \in C$, two Γ-equivalent points in U_ϵ are $\Gamma \cap P$-equivalent.

3. Siegel domain of the third kind

Definition 3.1. A *Siegel domain of the second kind* or a *Piatetski–Shapiro domain* is a domain D in $\mathbb{C}^m \times \mathbb{C}^n$ consisting of (z, w) such that for a given non-empty open convex cone $C \subseteq \mathbb{R}^m$ not containing a full line and C-valued positive Hermitian form F on $\mathbb{C}^n$, we have

$$\mathrm{Im}(z) - F(w, w) \in V\,.$$

A C-valued positive Hermitian form F on $\mathbb{C}^n$ is a $\mathbb{R}^m$-valued Hermitian form such that for non-zero $v \in \mathbb{C}^n$, $F(v, v) \in \overline{C} \setminus \{0\}$.

We need a even more general result. Fix a parameter space D, which is usually a bounded symmetric domain in $\mathbb{C}^k$ and a family of quasi-Hermitian forms (a sum of a Hermitian form and a symmetric form) L_t depending continuous on $t \in D$ with value in $\mathbb{R}^m$.

Definition 3.2. A *Siegel domain of the third kind* is the subset of $(z, w, t) \in \mathbb{C}^m \times \mathbb{C}^n \times \mathbb{C}^k$ such such that

$$\mathrm{Im}\, z - \mathrm{Re}\, F_t(w, w) \in C\,.$$

Why do Siegel domains of the third kind matter? This very fact relies on the whole theory we learned in the first part of the seminar. Roughly speaking, a bounded symmetric domain can be realized as a Siegel domain of the third kind if we fix a boundary component. Siegel domains of the third kind are simple, concrete and one can do explicit computations with these models, as we have seen in the toy model in Section 2.

3.1. Realization of bounded symmetric domains as Siegel domains of the third kind.

Let D be a simple bounded symmetric domain with a base point o and F be a boundary component of D. Take an $\mathbb{R}$-algebraic group $\mathcal{G}$ such that $\mathcal{G}(\mathbb{R})^+ = \mathrm{Aut}(D)^+$.

We recall a few notations introduced in the previous lectures. The normalizer of F is denoted by $\mathcal{N}(F)$. We have shown that it is representable and is a parabolic subgroup of $\mathcal{G}$. Let $\mathcal{W}(F)$ denote the unipotent radical of $\mathcal{N}(F)$. Let $w_F \in X_*(\mathcal{G})_\mathbb{R}$ be the cocharacter induced by F through the Harish-Chandra map: on $\mathbb{R}$-points it is

$$t \mapsto \mathrm{HC}_F\left(1; \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}\right)\,.$$

Then we have a canonical Levi decomposition

$$\mathcal{N}(F) = \mathcal{P}(w_F) = \mathcal{Z}(w_F) \ltimes \mathcal{W}(F)\,.$$

Set $\mathcal{U}(F) = [\mathcal{W}(F), \mathcal{W}(F)]$ in the sense of algebraic groups. Recall that $\mathcal{U}(F)$ as a unipotent group in char 0, its underlying real manifold can be identified with a Euclidean space. We write $U(F) = \mathcal{U}(F)(\mathbb{R})$ as a manifold. Let $\Omega_F \in U(F)$ be the canonical base point in $U(F)$ constructed from the Harish-Chandra map

$$\Omega_F := \mathrm{HC}_F\left(1; \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)\,.$$

The Levi part admits a canonical factorization

$$\mathcal{Z}(w_F)(\mathbb{R}) = G_h(F)G_\ell(F)M(F)$$

with $G_h(F)$ semi-simple without compact factors, $G_\ell(F)$ reductive without compact factors and $M(F)$ compact. Each factor has an underlying algebraic group.

Set $C(F)$ as the $G_\ell(F)$-orbit of Ω_F. Then $C(F) \subseteq U(F)$ is a symmetric cone.

There is a canonical $\mathcal{N}(F)(\mathbb{R})^+$-equivariant map $\Phi_F : D \to C(F)$ sending o to Ω_F so that as a real manifold

$$D \cong F \times C(F) \times W(F)$$

*There is a typo in [AMSRT10, Page 103] at this crucial place!

through $x \mapsto (\pi_F(x), \Phi_F(x), w(x))$. Recall that $p_F : D \to F$ is defined as sending $x \in D$ to $\lim_{t\to 0} w_F(t)x$. We will not recall the definition of $w(x)$ here.

Define

$$D(F) := U(F)_{\mathbb{C}} D \subseteq \check{D}\,.$$

This makes sense as $\check{D}$ is a quotient of $G_{\mathbb{C}}$. The projection Φ_F admits a canonical extension $\Phi_F : D(F) \to U(F)$. Moreover, D is exactly the set of $x \in D(F)$ with $\Phi_F(x) \in C(F)$.

sdasSiegdom **Theorem 3.3** (Korányi–Wolf, Piatetskii–Shapiro). *D is isomorphic to a Siegel domain of the third kind:*

$$\left\{(x, y, z) \in U(F)_{\mathbb{C}} \times \mathbb{C}^k \times F : \operatorname{Im} x - h_z(y, y) \in C(F)\right\}$$

for some $k \in \mathbb{N}$ and some family of semi-Hermitian forms h_z ($z \in F$) on $\mathbb{C}^k$ with value in $U(F)_{\mathbb{C}}$.

In the next few talks, we will see how the toy model in Section 2 can be extended to the compactifications of the Siegel domains in Theorem 3.3.

References

AMRT [AMSRT10] A. Ash, D. Mumford, P. Scholze, M. Rapoport, and Y. Tai. Smooth compactifications of locally symmetric varieties. Cambridge University Press, 2010.

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