NOTES ON SHIMURA VARIETIES II. HARISH-CHANDRA MAP

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Contents

1. Introduction

This is one of a series of notes prepared for a seminar on the toroidal compactifications of Shimura varieties.

In this note, we construct the Harish-Chandra map of a bounded symmetric domain. We add a number of details to the proof in [AMRT AMSRT10].

2. Harish-Chandra map

Let $D = G/K$ be a bounded symmetric domain. We fix a base point o corresponding to $1 \in G$. Recall that there is an $\mathbb{R}$-algebraic group $\mathcal{G}$ such that $G = \mathcal{G}(\mathbb{R})^+$. Let s be the $\mathbb{R}$-rank of $\mathcal{G}$.

2.1. **Harish-Chandra map.** The notations x_i, y_i, h_i $(i = 1, \ldots, s)$ correspond to $x_{\gamma_i}, y_{\gamma_i}, h_{\gamma_i}$ in Part I. Let us briefly recall the definitions. Recall that $\gamma_i \in \Delta$ is a well-chosen sequence of strongly orthogonal roots. The element h_i is

lma:sl2trip **Lemma 2.1.** *For each i,*
$$\left(-\frac{1}{2}y_i + \frac{1}{2}\mathrm{i}h_i, -\frac{1}{2}y_i - \frac{1}{2}\mathrm{i}h_i, x_i\right)$$
is a $\mathfrak{sl}_2$-triple. Moreover, these $\mathfrak{sl}_2$-triples for different i commute.

Proof. We compute
$$[x_i, y_i] = [X_{\gamma_i} + X_{-\gamma_i}, \mathrm{i}(X_{\gamma_i} - X_{-\gamma_i})] = -2\mathrm{i}[X_{\gamma_i}, X_{-\gamma_i}] = -2\mathrm{i}h_i\,.$$
Also
$$[x_i, h_i] = [X_{\gamma_i}, h_i] + [X_{-\gamma_i}, h_i] = -2X_{\gamma_i} + 2X_{-\gamma_i} = 2\mathrm{i}y_i\,.$$
And similarly,
$$[y_i, h_i] = -2\mathrm{i}x_i\,.$$
From these, it follows that the given triple is indeed a $\mathfrak{sl}_2$-triple.

The fact that different triples commute follows immediately from the strongly orthogonality of γ_i. $\square$

We introduce the map $h^{\mathrm{SL}} : U_1 \to \mathrm{SL}(2, \mathbb{R})$ by
$$h^{\mathrm{SL}}(e^{\mathrm{i}\theta}) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Recall that $u : U_1 \to G$ is Deligne's map introduced last time.

The following lemma is clear from the definition of u.

Date: April 12, 2022.

a:addeligne **Lemma 2.2.** *The adjoint action* $\operatorname{Ad} u(z)$ *fixes* h_i*, multiply* X_{γ_i} *by* z *and* $X_{-\gamma_i}$ *by* z^{-1}*.*

thm:HC **Theorem 2.3.** *There is a morphism* $\mathrm{HC} : U_1 \times \mathrm{SL}(2,\mathbb{R})^s \to G$ *such that*

(1) $\varphi(z, h^{\mathrm{SL}}(z), \ldots, h^{\mathrm{SL}}(z)) = u(z)^2$.
(2) The map

$$\mathrm{dHC} : \mathbb{R} \oplus \bigoplus_{i=1}^{s} \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}\,.$$

restricted to the second component is given by

$$\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{bmatrix}, \ldots, \begin{bmatrix} a_r & b_r \\ c_r & -a_r \end{bmatrix} \right) \mapsto \sum_{i=1}^{s} a_i x_i - \frac{b+c}{2} y_i + \frac{b-c}{2} \mathrm{i} h_i\,.$$

(3) The restriction of HC *to* $\mathrm{SL}(2,\mathbb{R})^s \to G$ *is algebraic.*

The map HC is known as the *Harish-Chandra map of D*.

The first step of the proof is omitted from the book [AMSRT10]. Longke Tang explained the argument to me.

Proof. **Step 1**. As a first step, we construct a homomorphism $\varphi_2 : \mathrm{SL}(2,\mathbb{R})^s \to G$ corresponding to the Lie algebra homomorphism in (2).

We first construct

$$\varphi_{2,\mathbb{C}} : \mathrm{SL}(2,\mathbb{C})^s \to G_{\mathbb{C}}\,.$$

This can be constructed by Lie correspondence and Lemma 2.1. By the general theory of semisimple Lie algebras [MO1], $\varphi_{2,\mathbb{C}}$ is algebraic. It follows from Galois descent that $\varphi_{2,\mathbb{C}}$ descends to an algebraic morphism

$$\mathrm{SL}_2^s \to \mathcal{G}$$

over $\mathbb{R}$, whose induced map on Lie algebras is given by the given map.

Step 2. We can now compute using Lemma 2.2:

$$\operatorname{Ad} u(z)^2|_{\mathfrak{g}'} = \operatorname{Ad} \varphi_2(h^{\mathrm{SL}}(z), \ldots, h^{\mathrm{SL}}(z))|_{\mathfrak{g}'}\,.$$

Here $\mathfrak{g}'$ is the Lie subalgebra of $\mathfrak{g}$ generated by $x_i, y_i, \mathrm{i}h_i$ $(i = 1, \ldots, s)$. By construction, both $u(z)^2$ and $\varphi_2(h^{\mathrm{SL}}(z), \ldots, h^{\mathrm{SL}}(z))$ lie in $\exp \mathfrak{h}$, so they commute. It follows that

$$u^2(z) = \varphi_1(z)\varphi_2(h^{\mathrm{SL}}(z), \ldots, h^{\mathrm{SL}}(z))$$

for some $\varphi_1 : U_1 \to G$ centralizing the image of φ_2. Define $\mathrm{HC} = \varphi_1 \cdot \varphi_2 : U_1 \times \mathrm{SL}(2,\mathbb{R})^s \to G$ finishes the proof. □

From the general theory of symmetric spaces, we find

Corollary 2.4. *The map* HC *induces a holomorphic symmetric map* $\tilde{\mathrm{HC}} : \mathbb{H}^s \to D$ *sending* $(\mathrm{i}, \ldots, \mathrm{i})$ *to* o*.*

We are interested in $\tilde{\mathrm{HC}}$ partially because of the following proposition.

Proposition 2.5 (Satake)**.** *Any symmetric holomorphic map* $\psi : \mathbb{H} \to D$ *with* $\psi(\mathrm{i}) = o$ *is of the form*

$$\psi(z) = k\tilde{\mathrm{HC}}(z_1, \ldots, z_r)$$

for some $k \in K$ *and* z_i *is either* z *or* i*.*

Proof. From the general theory, ψ can be lifted to a homomorphism

$$\Psi = \psi_1 \cdot \psi_2 : U^1 \times \mathrm{SL}(2,\mathbb{R}) \to G$$

with $u(z)^2 = \Psi_2(z, h^{\mathrm{SL}}(z))$.

As ψ_2 commutes with Cartan involution at i and o, we have

$$\mathrm{d}\psi_2 \left(\mathbb{R} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \subseteq \mathfrak{p}\,.$$

From Iwasawa decomposition, we know that $\mathfrak{p} = \mathrm{Ad}\, K \cdot \mathfrak{a}$, up to conjugating ψ, we may assume that

$$\mathrm{d}\psi_2 \left(\mathbb{R} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right) \subseteq \mathfrak{a}\,.$$

As ψ commutes with Deligne map, we have

$$\mathrm{d}\psi_2 \left(\mathbb{R} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) \subseteq J\mathfrak{a}\,.$$

Here J is the complex structure on $\mathfrak{p}$. It follows that

$$\mathrm{d}\psi_2 \left(\mathbb{R} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \subseteq [\mathfrak{a}, J\mathfrak{a}]\,.$$

In particular, Ψ factorizes through the image of HC and ψ_2 factorizes through $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SL}(2,\mathbb{R})^s$. But each component of such a map is either trivial or conjugate to identity by some element k_i in the maximal compact torus $\mathrm{SO}(2,\mathbb{R})$. We set $k_i = 1$ if the corresponding component is trivial. Then

$$\mathrm{HC}(1, k_1, \dots, k_r) \cdot \psi$$

has the desired form. ☐

3. Root systems

We use the same notations as the previous section.

We consider the $\mathbb{R}$-algebraic group $\mathcal{G}$. We have constructed a maximal $\mathbb{R}$-split torus $\mathcal{A}$ in $\mathcal{G}$ generated by the Lie algebra $\mathfrak{a}$. The next step is to understand the corresponding root system ${}_{\mathbb{R}}\Delta$. We refer to [Mil17, Section 25] for the general theory of root systems of non-split algebraic groups.

3.1. **Decomposition using Harish-Chandra map.** As a fist step, we will study the decomposition of $\mathfrak{g}$ under the adjoint action

$$\mathrm{Ad} \circ \mathrm{HC} : U_1 \times \mathrm{SL}(2,\mathbb{R})^s \to \mathrm{GL}(\mathfrak{g})\,.$$

We want to decompose $\mathfrak{g}$ into irreducible pieces.

As is well-known, the irreducible representations of a product group are tensor products of irreducible representations of each factor. The irreducible representations of U_1 are

(1) $V_0 = \mathbb{R}$ with the trivial action.
(2) $V_k = \mathbb{R}^2$ ($k \in \mathbb{Z}_{>0}$) with

$$U_1 \ni z \mapsto h^{\mathrm{SL}}(z^k)\,.$$

Note that both representations are algebraic.

Similarly, the irreducible representations of $\mathrm{SL}(2,\mathbb{R})$ are

(1) $W_k = \mathrm{Sym}^k\, \mathbb{R}^2$ ($k \in \mathbb{N}$), where $\mathbb{R}^2$ is the standard representation of $\mathrm{SL}(2,\mathbb{R})$.

Again, all of them are algebraic representations.

So an irreducible representation of $U_1 \times \mathrm{SL}(2,\mathbb{R})^s$ takes the form

$$V_i \otimes W_{j_1} \otimes \cdots \otimes W_{j_s}\,.$$

Only some of them can happen in $\mathfrak{g}$. Recall that $\mathrm{HC}(z; h^{\mathrm{SL}}(z), \dots, h^{\mathrm{SL}}(z)) = u(z)^2$. It is well-known that Deligne's map u has weight $0, \pm 1$. It follows that

$$i + j_1 + \cdots + j_s = 2 \text{ or } 0\,.$$

Proposition 3.1. *The irreducible representations of $U_1 \times \mathrm{SL}(2,\mathbb{R})^s$ that occurs in $\mathfrak{g}$ are*

(a) $V_0 \otimes (W_0 \otimes \cdots \otimes W_2 \otimes \cdots \otimes W_0)$.
(b) $V_0 \otimes (W_0 \otimes \cdots \otimes W_1 \otimes \cdots \otimes W_1 \otimes \cdots \otimes W_0)$.
(c) $V_1 \otimes (W_0 \otimes \cdots \otimes W_1 \otimes \cdots \otimes W_0)$.
(e) $V_0 \otimes (W_0 \otimes \cdots \otimes W_0)$.

Proof. The only point is that we have excluded the possibility of $V_2 \otimes (W_0 \otimes \cdots \otimes W_0)$. Let $V \subseteq \mathfrak{g}$ be such a factor. Then u^2 acts on V by the representation V_2. In particular, $V \subseteq \mathfrak{p}$.

In this case, also observe that $\mathfrak{a}$ acts trivially on V, but $\mathfrak{a}$ is a maximal abelian subalgebra of $\mathfrak{p}$, so $V \subseteq \mathfrak{a}$. But $\mathrm{SL}(2,\mathbb{R})^s$ does not act trivially on any subspace of $\mathfrak{a}$. $\square$

By a similar argument, type (a) occurs s-times, each corresponding to the image of one $\mathfrak{sl}(2,\mathbb{R})$ factor under dHC.

As a consequence, we get information of the complex root decomposition of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Recall that

$$h_i = -\mathrm{idHC}_{\mathbb{C}}\left(0; 0, \ldots, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \ldots, 0\right) .$$

But we know how h_i acts on each factor.

Type (a): Take the standard basis (h_i, x_i, y_i) for the factor of type (a) corresponding to the i-th factor. By the proof of Lemma 2.1,

$$[h_i, x_i] = 2\mathrm{i}y_i\,, \quad [h_i, y_i] = 2\mathrm{i}x_i\,.$$

The root spaces are

$$\mathfrak{g}^{\gamma_i} = \mathbb{C}X_i\,, \quad \mathfrak{g}^{-\gamma_i} = \mathbb{C}X_{-i}\,.$$

Type (b): We place W_1 at the $i < j$-places. Take the standard basis e_1, e_2 and f_1, f_2 for the two W_1 factors.

We only have to consider the i, j-th factors. So the problem reduces to compute the roots of the representation $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ on $W_1 \otimes W_1$. This further reduces to compute the weights of the standard representation of $\mathrm{SL}(2,\mathbb{R})$. It is well-known that such a representation has two weights, in our case represented by $\pm\frac{1}{2}\gamma_i$. So for type (b), the complex roots are

$$\mathfrak{g}^{\pm\frac{1}{2}\gamma_i \pm \frac{1}{2}\gamma_j} = \mathbb{C}\begin{bmatrix} 1 \\ \pm\mathrm{i} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \pm\mathrm{i} \end{bmatrix} .$$

The two $\pm$ signs on the LHS correspond to those two on the RHS.

Type (c): We place W_1 at the i-th place. Then we need to compute the roots of $U_1 \times \mathrm{SL}(2,\mathbb{R})$ acting on $V_1 \times W_1$. The latter factor can be handled as before. The former factor is easy: let $\pm\mu$ denote the two weights of U_1 on V_1. Then for type (c), the complex weights are

$$\mathfrak{g}^{\pm\frac{1}{2}\mu \pm \frac{1}{2}\gamma_i} = \mathbb{C}\begin{bmatrix} 1 \\ \pm\mathrm{i} \end{bmatrix} \otimes \begin{bmatrix} 1 \\ \pm\mathrm{i} \end{bmatrix} .$$

However, we are only interested in the roots of G, so we should discard the μ-factors:

$$\mathfrak{g}^{\pm\frac{1}{2}\gamma_i} = V_1 \otimes \begin{bmatrix} 1 \\ \pm\mathrm{i} \end{bmatrix} .$$

Type (e): Completely parallel to Type (c), we find

$$\mathfrak{g}^0 = V_0\,.$$

3.2. **Real root system.** Now we want to understand the real root decomposition of $(\mathfrak{g}, \mathfrak{a})$. Namely,

q:realroot}

$$\mathfrak{g} = Z_{\mathfrak{g}}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in {}_{\mathbb{R}}\Delta} \mathfrak{g}^{\alpha}\,, \tag{3.1}$$

where $\mathfrak{g}^{\alpha}$ is the subspace of $\mathfrak{g}$ where $\mathfrak{a}$ acts through α.

From the general theory, such (the complexification) a decomposition can be obtained from the weight decomposition of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}})$, but what is actually under our disposal is the weight decomposition of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$.

By a simple trick that we will explain later, these decompositions are in fact equivalent.

Define

$$\mathfrak{a}' = \sum_{i=1}^{s} \mathbb{R} \cdot h_i \subseteq \mathrm{i}\mathfrak{h}\,.$$

Let ${}_{\mathbb{R}}\Delta'$ be the set of non-zero linear maps $\mathfrak{a}' \to \mathbb{R}$ given by restricting elements in Δ to $\mathfrak{a}'$.

We then have a decomposition

realroota'}

$$\mathfrak{g}_{\mathbb{C}} = Z_{\mathfrak{g}_{\mathbb{C}}}(\mathfrak{a}') \oplus \bigoplus_{\alpha \in {}_{\mathbb{R}}\Delta'} \mathfrak{g}^{\alpha}\,, \quad \mathfrak{g}^{\alpha} = \bigoplus_{\eta \in \Delta, \eta|_{\mathfrak{a}'} = \alpha} \mathfrak{g}^{\eta}\,. \tag{3.2}$$

We study the following transform. Define

$$c = \mathrm{HC}_{\mathbb{C}}\left(1; \frac{1}{\sqrt{2}}\begin{bmatrix}1 & \mathrm{i}\\ \mathrm{i} & 1\end{bmatrix}, \dots, \frac{1}{\sqrt{2}}\begin{bmatrix}1 & \mathrm{i}\\ \mathrm{i} & 1\end{bmatrix}\right) \in G_{\mathbb{C}}\,.$$

prop:adca

Proposition 3.2. *We have*

$$\mathrm{Ad}(c)(\mathfrak{a}) = \mathfrak{a}'\,.$$

Proof. A direct computation shows that

$$\frac{1}{\sqrt{2}}\begin{bmatrix}1 & \mathrm{i}\\ \mathrm{i} & 1\end{bmatrix} = \exp\begin{bmatrix}0 & \mathrm{i}\frac{\pi}{4}\\ \mathrm{i}\frac{\pi}{4} & 0\end{bmatrix}.$$

We compute

$$\mathrm{Ad}(c)(x_i) = \exp \mathrm{ad}\, \mathrm{dHC}_{\mathbb{C}}\left(0; \begin{bmatrix}0 & \mathrm{i}\frac{\pi}{4}\\ \mathrm{i}\frac{\pi}{4} & 0\end{bmatrix}, \dots, \begin{bmatrix}0 & \mathrm{i}\frac{\pi}{4}\\ \mathrm{i}\frac{\pi}{4} & 0\end{bmatrix}\right)(x_i) = \exp \mathrm{ad}(-\frac{\pi}{4}\mathrm{i}y_i)(x_i)\,.$$

By the computations in the proof of Lemma 2.1, we find easily that this is equal to $-h_i$. ☐

From this result, the decomposition (3.2) is just the complexification of the $\mathrm{Ad}(c)$-image of the real root decomposition (3.1).

Before stating the main theorem, let us introduce one more notation. From the above description, we know that the real root spaces can be grouped into four categories (a), (b), (c), (e), obtained from conjugating the corresponding factors in (3.2). So in particular, (a) factors occurs $2s$-times. We call the corresponding roots $\pm\beta_i \in {}_{\mathbb{R}}\Delta$. The normalization is $\beta_i(x_j) = 2\delta_{ij}$.

thm:root

Theorem 3.3. *Assume that D is irreducible. Then the set ${}_{\mathbb{R}}\Delta$ has two possibilities:*

C_s

$$\left\{\pm\frac{1}{2}(\beta_i + \beta_j) \text{ for } i \geq j; \pm\frac{1}{2}(\beta_i - \beta_j) \text{ for } i > j\right\};$$

BC_s

$$\left\{\pm\frac{1}{2}(\beta_i + \beta_j) \text{ for } i \geq j; \pm\frac{1}{2}(\beta_i - \beta_j) \text{ for } i > j; \pm\frac{1}{2}\beta_i\right\}.$$

In both cases, the Weyl group is $\gamma_i \mapsto \pm\gamma_{\sigma i}$ for some $\sigma \in \mathcal{S}_s$.

Take a minimal parabolic of $\mathcal{G}$ whose Lie algebra contains $\mathfrak{a}$, up to the action of the Weyl group, we may assume that $\gamma_1 > \cdots > \gamma_s$. So the simple roots are

C_s $\left\{\frac{1}{2}(\beta_i - \beta_{i+1}) \text{ for } i = 1, \dots, s-1; \beta_s\right\}$.

BC_s $\left\{\frac{1}{2}(\beta_i - \beta_{i+1}) \text{ for } i = 1, \dots, s-1; \frac{1}{2}\beta_s\right\}$.

We omit the straightforward proof, see [AMSRT10, Proposition 2.8].

3.3. **Induced maps.** Now let $S \subseteq \{1, \dots, s\}$ be a subset. We define a map using Harish-Chandra map:

$$\mathrm{HC}_S : U_1 \times \mathrm{SL}(2, \mathbb{R}) \to G\,, \quad (z, x) \mapsto \mathrm{HC}(z; \dots, z \dots, x \dots)\,,$$

where we insert z for places $i \notin S$ and x for places $i \in S$.

From the general theory of symmetric spaces, we find a symmetric map $\mathbb{H} \to D$ induced by HC_S and a symmetric extension $\mathbb{P}^1 \to \check{D}$, uniquely determined by $\mathrm{i} \mapsto o$.

Proposition 3.4. *The image of ∞ under $\mathbb{P}^1 \to \check{D}$ is $\sum_{i \in S} X_{\gamma_i} \in \mathfrak{p}_+$.*

Proof. Observe that
$$\infty = \lim_{t\to\infty} \exp t \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \mathrm{i}\,.$$
So the image of ∞ is given by
$$\lim_{t\to\infty} \exp t \sum_{i\in S} x_i \cdot o = \sum_{i\in S} X_{\gamma_i}$$
from the explicit formula in Part I. □

We define a cocharacter $w_S : \mathbb{G}_m \to \mathcal{G}$ by
$$w_S(t) = \mathrm{HC}_S\left(1, \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}\right)\,.$$

prop:dw1 **Proposition 3.5.** *w_S is a well-defined cocharacter of $\mathcal{G}$. Moreover, $\mathrm{d}w_S : \mathbb{R} \to \mathfrak{g}$ is given by*
$$\mathrm{d}w_S(1) = \sum_{i\in S} x_i\,.$$

Proof. That w_S defines an algebraic cocharacter follows from Theorem 2.3. The differential of w_S is computed using Theorem 2.3. □

Next we consider the parabolic group associated to w_S. We let
$$\mathcal{P}_S = \mathcal{P}_{\mathcal{G}}(w_S) \subseteq \mathcal{G}\,.$$
We have the real root decomposition
$$\mathfrak{p}_S = Z_{\mathfrak{g}}(\mathfrak{a}) \oplus \bigoplus_{\alpha\in{}_{\mathbb{R}}\Delta, (\alpha, \mathrm{d}w_S)\geq 0} \mathfrak{g}^\alpha\,.$$

Lemma 3.6. *The set of roots $\{\alpha \in {}_{\mathbb{R}}\Delta, (\alpha, \mathrm{d}w_S) \geq 0\}$ is given by*
$$\left\{\frac{1}{2}(\pm\beta_i \pm \beta_j), \pm\frac{1}{2}\beta_i\,, i, j \notin S\right\} \cup \left\{\frac{1}{2}(\pm\beta_i \pm \beta_j), \pm\frac{1}{2}\beta_i, i \in S, j = 1, \ldots, s\right\}\,.$$

Proof. This is an immediate consequence of Proposition 3.5 and Theorem 3.3. □

References

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