NOTES ON SHIMURA VARIETIES III. STANDARD BOUNDARY COMPONENTS

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Contents

1. Introduction

This is one of a series of notes prepared for a seminar on the toroidal compactifications of Shimura varieties.

In this part, we study the standard boundary components. In the next part, we will prove that the information of standard boundary components actually implies the information of all boundary components.

2. Summary of results from previous lectures

sec:sum

Consider an irreducible bounded symmetric domain $D = G/K$, where $G = \mathcal{G}(\mathbb{R})^+$ for some semisimple adjoint $\mathbb{R}$-algebraic group $\mathcal{G}$, K is a maximal compact subgroup corresponding to $o \in D$. We have a corresponding Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}\,.$$

Choose a Cartan subalgebra $\mathfrak{h}$ and a suitable ordering on $\mathrm{i}\mathfrak{h}^\vee$, we get a further decomposition

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_+ \oplus \mathfrak{p}_-\,.$$

A root $\alpha \in \Phi = \Phi(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is compact (resp. positive, negative) if $\mathfrak{g}^\alpha$ is in $\mathfrak{k}$ (resp. $\mathfrak{p}_+$, $\mathfrak{p}_-$). We choose a maximal set of strongly orthogonal positive roots $\gamma_1, \ldots, \gamma_r$ and $X_i \in \mathfrak{g}^{\gamma_i}$, $X_{-i} \in \mathfrak{g}^{-\gamma_i}$, $h_i \in \mathrm{i}\mathfrak{h}$ such that

$$X_i - X_{-i}, \mathrm{i}(X_i + X_{-i}) \in \mathfrak{k} + \mathrm{i}\mathfrak{p}\,, \quad [X_i, X_{-i}] = h_i\,.$$

Let

$$x_i = X_i + X_{-i}\,, \quad y_i = \mathrm{i}(X_i - X_{-i})\,.$$

Set

$$\mathfrak{a} = \sum_{i=1}^r \mathbb{R} x_i\,.$$

Then $\mathfrak{a}$ is a maximal Abelian subspace of $\mathfrak{p}$ and it is the Lie algebra of a maximal split torus in $\mathcal{G}$.

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Let $\beta_1, \dots, \beta_r$ be the basis of $\mathfrak{a}^\vee$ dual to $x_1/2, \dots, x_r/2$. We take the lexicographic ordering on $\mathfrak{a}^\vee$ defined by the basis $X_1, \dots, X_r$ (so that $\beta_1 > \cdots > \beta_r$). Then the simple $\mathbb{R}$-roots in ${}_{\mathbb{R}}\Phi = {}_{\mathbb{R}}\Phi(\mathfrak{g}, \mathfrak{a})$ is

$$\alpha_i = \frac{1}{2}(\beta_i - \beta_{i+1}) \quad (i = 1, \dots, r-1), \alpha_r ,$$

where $\alpha_r = \gamma_r$ if ${}_{\mathbb{R}}\Phi$ has type C_r and $\alpha_r = \frac{1}{2}\gamma_r$ if ${}_{\mathbb{R}}\Phi$ has type BC_r. This numbering of simple roots is known as the *canonical numbering* in [BB66].

We summarize the situation in the following proposition:

prop:roots **Proposition 2.1.** *Under the assumptions above, the set of simple roots is*

C_r *In this case,*

$$\left\{\alpha_i = \frac{1}{2}(\beta_i - \beta_{i+1}) \text{ for } i = 1, \dots, r-1; \alpha_r = \beta_r\right\} .$$

BC_r *In this case,*

$$\left\{\alpha_i = \frac{1}{2}(\beta_i - \beta_{i+1}) \text{ for } i = 1, \dots, r-1; \alpha_r = \frac{1}{2}\beta_r\right\} .$$

The full set of roots ${}_{\mathbb{R}}\Phi$ *is*

C_r *In this case,*

$$\left\{\pm\frac{1}{2}(\beta_i \pm \beta_j) \text{ for } 1 \leq i < j \leq r; \pm\beta_i \text{ for } i = 1, \dots, r\right\} .$$

BC_r *In this case,*

$$\left\{\pm\frac{1}{2}(\beta_i \pm \beta_j) \text{ for } 1 \leq i < j \leq r; \pm\beta_i \text{ for } i = 1, \dots, r; \pm\frac{1}{2}\beta_i \text{ for } i = 1, \dots, r\right\} .$$

We will refer to these explicit roots for concrete computations.

3. Parabolic subgroups of reductive groups

We will work in the fairly general setting here. Let k be an arbitrary field, of any characteristic. Recall that a *connected* smooth algebraic group $\mathcal{G}$ over k is reductive if $\mathcal{R}_u(\mathcal{G}_{k^a})$ is trivial, where k^a is an algebraic closure of k.

3.1. Standard parabolic subgroups.

Fix a reductive group $\mathcal{G}$ over k.

We will recall the basic facts regarding parabolic subgroups of $\mathcal{G}$. When there exists a Borel subgroup, the whole theory is well-known, covered by any standard textbooks. We will need a more general case.

Theorem 3.1. *All maximal split k-tori are $\mathcal{G}(k)$-conjugate. All minimal parabolic k-subgroups of $\mathcal{G}$ are $\mathcal{G}(k)$-conjugate.*

Any parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$ contains a maximal split torus and these maximal split tori are all $\mathcal{P}(k)$-conjugate.

All parabolic subgroups are self-normalizing and connected.

Let $\mathcal{S}$ be a fixed maximal split torus of $\mathcal{G}$. We will try to classify parabolic subgroups containing $\mathcal{S}$.

Theorem 3.2. *There is a bijection between the set of minimal parabolic subgroups of $\mathcal{G}$ containing $\mathcal{S}$ and the set of positive systems of roots in $\Phi = {}_k\Phi(\mathcal{G}, \mathcal{S})$. More explicitly, given a minimal parabolic subgroup $\mathcal{P}$, the corresponding positive system is just $\Phi(\mathcal{P}, \mathcal{S})$.*

Definition 3.3. A subset Ψ of a root system Φ is *paraboic* if

(1) Ψ is closed: if $a, b \in \Psi$, $a + b \in \Phi$, then $a + b \in \Psi$.
(2) $\Psi \cup (-\Psi) = \Phi$.

Theorem 3.4. *Fix a maximal split torus $\mathcal{S}$ in $\mathcal{G}$. Let $\Phi = \Phi(\mathcal{G}, \mathcal{S})$. There is a bijection between the set of parabolic subgroups of $\mathcal{G}$ containing $\mathcal{S}$ and the set of parabolic subsets of Φ. More explicitly, given a parabolic subgroup $\mathcal{P}$, the corresponding positive system is just $\Phi(\mathcal{P}, \mathcal{S})$.*

Next we fix a minimal parabolic subgroup $\mathcal{P}_0$ of $\mathcal{G}$ containing $\mathcal{P}_0$. We say a parabolic subgroup of $\mathcal{G}$ is *standard* (with respect to $\mathcal{P}_0$) if it contains $\mathcal{P}_0$. This notion is useful because

thm:uni **Theorem 3.5.** *Every $\mathcal{G}(k)$-conjugacy class of parabolic subgroups of $\mathcal{G}$ contains a unique member containing $\mathcal{P}_0$.*

r:bijjparpar **Corollary 3.6.** *There is a bijection between the set of standard parabolic subgroups of $\mathcal{G}$ with respect to $\mathcal{P}_0$ and parabolic subsets of Φ containing $\Phi^+ := \Phi(\mathcal{P}_0, \mathcal{S})$.*

Let Δ be the simple roots associated to the positive system Φ^+. Then the set of parabolic subsets of Φ containing Φ^+ is in bijection with subsets of $I \subseteq \Delta$. More explicitly, given $I\Delta$, the parabolic subset if

$$\Phi^+ \cup (\Phi \cap \mathbb{Z} \cdot I)\,.$$

3.2. **Dynamic theory.** We review the dynamic theory of general algebraic groups. Our reference is [Mil17, Section 13.d].

Let $\mathcal{G}$ be a smooth algebraic group over a field k and $\lambda : \mathbb{G}_m \to \mathcal{G}$ a cocharacter (1-parameter subgroup) of $\mathcal{G}$. Let

$$\mathcal{P}_{\mathcal{G}}(\lambda)(R) := \left\{ g \in \mathcal{G}(R) : \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} \text{ exists} \right\}$$

for any (small) k-algebra R. Then $\mathcal{P}_{\mathcal{G}}(\lambda)$ is an algebraic subgroup of $\mathcal{G}$.

Define $\mathcal{Z}_{\mathcal{G}}(\lambda) = \mathcal{Z}_{\mathcal{G}}(\lambda\mathbb{G}_m)$. Then in fact,

$$\mathcal{Z}_{\mathcal{G}}(\lambda) = \mathcal{P}_{\mathcal{G}}(\lambda) \cap \mathcal{P}_{\mathcal{G}}(-\lambda)\,.$$

Define

$$\mathcal{U}_{\mathcal{G}}(\lambda)(R) := \left\{ g \in \mathcal{P}_{\mathcal{G}}(\lambda)(R) : \lim_{t\to 0} \lambda(t) g \lambda(t)^{-1} = 1 \right\}\,.$$

Then $\mathcal{U}_{\mathcal{G}}(\lambda)$ is represented by a normal subgroup of $\mathcal{P}_{\mathcal{G}}(\lambda)$.

When $\mathcal{G}$ is connected, $\mathcal{Z}_{\mathcal{G}}(\lambda)$, $\mathcal{U}_{\mathcal{G}}(\lambda)$ and $\mathcal{P}_{\mathcal{G}}(\lambda)$ are all connected. We will assume that $\mathcal{G}$ is smooth from now on. We have the decomposition

{eq:PZU} (3.1)
$$\mathcal{P}_{\mathcal{G}}(\lambda) = \mathcal{Z}_{\mathcal{G}}(\lambda) \ltimes \mathcal{U}_{\mathcal{G}}(\lambda)\,.$$

We decompose

$$\mathfrak{g} = \bigoplus_{n\in\mathbb{Z}} \mathfrak{g}_n$$

according to the weights of $\mathrm{Ad}\circ\lambda$. Then

{eq:LiePUZ} (3.2)
$$\mathrm{Lie}\,\mathcal{P}_{\mathcal{G}}(\lambda) = \bigoplus_{n\geq 0} \mathfrak{g}_n\,, \quad \mathrm{Lie}\,\mathcal{U}_{\mathcal{G}}(\lambda) = \bigoplus_{n>0} \mathfrak{g}_n\,, \quad \mathrm{Lie}\,\mathcal{Z}_{\mathcal{G}}(\lambda) = \mathfrak{g}_0\,.$$

3.3. **Iwasawa and Langlands decompositions.** We will consider the following situation: let $\mathcal{G}$ be a semi-simple adjoint $\mathbb{R}$-algebraic group, $\mathcal{P}$ be a minimal parabolic subgroup of $\mathcal{G}$ and $\mathcal{A}$ be a maximal split torus contained in $\mathcal{P}$. Write $\mathcal{N} = \mathcal{R}_u\mathcal{P}$. Let $G = \mathcal{G}(\mathbb{R})^+$ and $N = \mathcal{N}(\mathbb{R})$. Take K to be the maximal compact torus of G orthogonal to $A = \mathcal{A}(\mathbb{R})$. We have the *Iwasawa decomposition*

$$G = KAN\,.$$

Definition 3.7. A *parabolic subgroup* of the real Lie group G is a subgroup of G of the form $G \cap \mathcal{P}'(\mathbb{R})$ for some parabolic subgroup $\mathcal{P}'$ of $\mathcal{G}$.

Write $P = \mathcal{P}(\mathbb{R}) \cap G$ the parabolic subgroup of G defined by $\mathcal{P}$. Let $M = Z_K(A)$, then we have the *Langlands decomposition*

$$P = MAN\,.$$

It is well-known that $Z_G(A) = MA$. One can also restate the Langlands decomposition in terms of algebraic groups.

4. Lie algebra computations

Now we come back to our notations in Section 2. In particular, We write $\alpha_1, \ldots, \alpha_r \in \mathfrak{a}^\vee$ for the simple roots with the canonical ordering.

4.1. **Standard maximal parabolic subgroups.** Let $\mathcal{A}$ be the maximal split torus of $\mathcal{G}$ with Lie algebra $\mathfrak{a}$. Take a minimal parabolic subgroup $\mathcal{P}$ of $\mathcal{G}$ containing $\mathcal{A}$ such that

$$
{}_{\mathbb{R}}\Phi(\mathcal{P},\mathcal{A}) = \Phi \cap \Big(\bigoplus_{i=1}^{r} \mathbb{N}\alpha_i\Big)\,.
$$

Also recall that we have the root decomposition,

$$
\mathfrak{g} = Z_{\mathfrak{g}}(\mathfrak{a}) \oplus \bigoplus_{\alpha \in {}_{\mathbb{R}}\Phi} \mathfrak{g}^{\alpha}\,.
$$

Observe that $Z_{\mathfrak{g}}(\mathfrak{a}) = \mathfrak{m} \oplus \mathfrak{a}$, where $\mathfrak{m}$ is the Lie algebra of the anisotropic part in the Langlands decomposition.

Definition 4.1. For $b = 1,\ldots,r$, let $\mathfrak{a}_b \subseteq \mathfrak{a}$ be the one-dimensional subspace on which all simple $\mathbb{R}$-roots but α_b vanishes:

$$
\mathfrak{a}_b := \{a \in \mathfrak{a} : \alpha_i(a) = 0 \text{ for } i = 1,\ldots,r, i \neq b\}\,.
$$

Proposition 4.2. *In both C_r and BC_r cases, if $b < s$,*

$$
\mathfrak{a}_b = \{\beta_1 = \cdots = \beta_b; \beta_{b+1} = \cdots = \beta_r = 0\}\,.
$$

If $b = s$,

$$
\mathfrak{a}_b = \{\beta_1 = \cdots = \beta_r\}\,.
$$

In other words, for any $b = 1,\ldots,r$,

{eq:ab}

$$
\mathfrak{a}_b = \mathbb{R}(x_1 + \cdots + x_b)\,. \tag{4.1}
$$

From now on, we fix $b = 1,\ldots,r$. Let $\mathcal{A}_b$ be the algebraic subgroup of $\mathcal{A}$ generated by $\mathfrak{a}_b$. We let $w_b \in \mathfrak{a}_b$ be the element corresponding to $x_1 + \cdots + x_b$.

We will construct a family of standard maximal parabolic subgroups $\mathcal{P}_b$ exhausting all $\mathcal{G}(\mathbb{R})$-conjugacy classes.

From the dynamic theory,

$$
\mathcal{P}_b = \mathcal{P}_{\mathcal{G}}(w_b)\,. \tag{4.2}
$$

More explicitly,

$$
\mathcal{P}_b = Z_{\mathcal{G}}(\mathcal{A}_b) \cdot \mathcal{N}\,.
$$

Proposition 4.3. *$\mathcal{P}_1,\ldots,\mathcal{P}_r$ are all maximal standard parabolic subgroups of $\mathcal{G}$.*

Proof. Observe that

$$
\{w_b \geq 0\} = \Phi \cap \left(\bigoplus_{i\neq b} \mathbb{Z}\alpha_i \oplus \mathbb{N}\alpha_b\right) \subseteq X^*(\mathcal{A})\,.
$$

From Corollary 3.6, it is obvious that $\mathcal{P}_1,\ldots,\mathcal{P}_r$ are all standard maximal parabolic subgroups. From Theorem 3.5, they are not conjugate to each other. On the other hand, there are exactly r-different standard maximal parabolic subgroups, it follows that we have all of them. □

Let

$$
\mathcal{W}_b = \mathcal{R}_u(\mathcal{P}_b) \subseteq \mathcal{N}\,.
$$

Then

q:Pbdecomp}

$$
\mathcal{P}_b = Z_{\mathcal{G}}(\mathcal{A}_b) \ltimes \mathcal{W}_b\,. \tag{4.3}
$$

From the dynamic theory,

$$
\mathcal{W}_b = \mathcal{U}_{\mathcal{G}}(w_b)\,.
$$

Proposition 4.4. *The set $\{w_b > 0\}$ is*

C_r

$$
\frac{1}{2}(\beta_i \pm \beta_j) \text{ for } i \leq b < j; \quad \frac{1}{2}(\beta_i + \beta_j) \text{ for } i \leq j \leq b\,.
$$

BC_r

$$
\frac{1}{2}(\beta_i \pm \beta_j) \text{ for } i \leq b < j; \quad \frac{1}{2}(\beta_i + \beta_j) \text{ for } i \leq j \leq b; \quad \frac{1}{2}\beta_i \text{ for } i \leq b\,.
$$

The sum of the corresponding root spaces is $\mathfrak{w}_b$ by (eq:LiePUZ 3.2).
Similarly,

prop:pb **Proposition 4.5.** *The Lie algebra $\mathfrak{p}_b$ can be described as*

(4.4)
$$\left\{\pm\frac{1}{2}(\pm\beta_i \pm \beta_j) \text{ for } b < i \le j \le r; \pm\frac{1}{2}\beta_i \text{ for } b < i \le r; \frac{1}{2}(\beta_i \pm \beta_j) \text{ for } 1 \le i \le b, 1 \le j \le r; \frac{1}{2}\beta_i \text{ for } 1 \le i \le b\right\}$$

in both the C_r case and the BC_r case.

Both propositions follow from (eq:LiePUZ 3.2).

4.2. **The weight filtration.** The following lemma is quite obvious:

Lemma 4.6. *We have*

$$(w_b, \beta_i) = \begin{cases} 2^{*}, & i \le b, \\ 0, & i > b. \end{cases}$$

The weights of each kind of root $\alpha \in {}_{\mathbb{R}}\Phi$ with respect to $\mathrm{d}w_b$ are:

(1) $\alpha = \pm\frac{1}{2}(\beta_i + \beta_j)$, $i, j \le b$, weight ± 2.
(2) $\alpha = \pm\frac{1}{2}(\beta_i - \beta_j)$, $i, j \le b$, weight 0.
(3) $\alpha = \pm\frac{1}{2}\beta_i$, $i \le b$, weight ± 1.
(4) $\alpha = \pm\frac{1}{2}(\beta_i + \beta_j)$, $i \le b$, $j > b$, weight ± 1.
(5) $\alpha = \pm\frac{1}{2}(\beta_i - \beta_j)$, $i \le b$, $j > b$, weight ± 1.
(6) $\alpha = \pm\frac{1}{2}\beta_i$, $i > b$, weight 0.
(7) $\alpha = \frac{1}{2}(\pm\beta_i \pm \beta_j)$, $i, j > b$, weight 0.

In particular, if we write $W_\bullet$ for the weight filtration on $\mathfrak{g}$ defined by w_b, the only possible weights are $-2, -1, 0, 1, 2$. We write $\mathfrak{g}^a = \mathrm{Gr}^W_a(\mathfrak{g})$.

Observe that

$$\begin{aligned} \mathfrak{p}_b &= \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^{0}, \\ \mathfrak{w}_b &= \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1}, \\ Z(\mathfrak{w}_b) &= \mathfrak{g}^{-2}, \\ Z_{\mathfrak{g}}(\mathfrak{a}_b) &= \mathfrak{g}^{0}. \end{aligned} \tag{4.5}$$

4.3. **A refinement of the decomposition.** In this section, we refine the decomposition (eq:Pbdecomp 4.3):

$$\mathcal{P}_b = Z_{\mathcal{G}}(\mathcal{A}_b) \ltimes \mathcal{W}_b$$

by further decomposition the Lie algebra $Z_{\mathfrak{g}}(\mathfrak{a}_b)$ of $Z_{\mathcal{G}}(\mathcal{A}_b)$. We will do the same thing at the level of algebraic groups in the next section.

Write

{eq:lb}
$$\mathfrak{l}_b = \sum_{\alpha \in {}_{\mathbb{R}}\Phi, \alpha = \sum_{i > b} a_i \alpha_i} \mathfrak{g}^\alpha + [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] = \sum_{\alpha \in {}_{\mathbb{R}}\Phi, \alpha = \sum_{i > b} a_i \beta_i} \mathfrak{g}^\alpha + [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]. \tag{4.6}$$

Similarly,

$$\mathfrak{l}'_b = \sum_{\alpha \in {}_{\mathbb{R}}\Phi, \alpha = \sum_{i < b} a_i \alpha_i} \mathfrak{g}^\alpha + [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}].$$

From the strong orthogonality of β_i, we easily see that the $\mathfrak{l}_b$'s are ideals in $Z_{\mathfrak{g}}(\mathfrak{a}_b)$.

We observe that $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] \in Z_{\mathfrak{g}}(\mathfrak{a}_b) = \mathfrak{m} \oplus \mathfrak{a}$. From Proposition 3.2 of the previous part, $X_i \in [\mathfrak{g}^{\alpha_i}, \mathfrak{g}^{-\alpha_i}]$. In fact, from the same proposition,

$$[\mathfrak{g}^{\alpha_i}, \mathfrak{g}^{-\alpha_i}] \cap \mathfrak{a} = \mathbb{R}X_i.$$

With respect to the root decomposition $(\mathfrak{g}, \mathfrak{a})$,

{eq:Zgab}
$$Z_{\mathfrak{g}}(\mathfrak{a}_b) = \mathfrak{m}_b \oplus \mathfrak{a}_b \oplus \mathfrak{l}'_b \oplus \mathfrak{l}_b, \tag{4.7}$$

where $\mathfrak{m}_b = \mathfrak{m} \cap Z_{\mathfrak{g}}(\mathfrak{a}_b)$.

It follows from (3.2) that

$$\mathfrak{p}_b = Z_{\mathfrak{g}}(\mathfrak{a}_b) \oplus \mathfrak{w}_b = \mathfrak{m}_b \oplus \mathfrak{a}_b \oplus \mathfrak{l}_b \oplus \mathfrak{l}'_b \oplus \mathfrak{w}_b\,. \tag{4.8}$$

Finally, observe that

Proposition 4.7. *$\mathfrak{l}_b$ is semi-simple.*

Proof. It suffices to show that $[\mathfrak{l}_b, \mathfrak{l}_b] = \mathfrak{l}_b$. The only non-trivial part is

$$\mathfrak{g}^{\beta_i} \subseteq [\mathfrak{l}_b, \mathfrak{l}_b]$$

whenever $i > b$. But we know that $x_i \in [\mathfrak{g}^{\alpha_i}, \mathfrak{g}^{-\alpha_i}]$, so any $X \in \mathfrak{g}^{\beta_i}$ can be written as

$$X = \frac{1}{2}[x_i, X]\,.$$

□

We observe that $\mathfrak{l}_b$ is a direct sum of Lie subalgebras of type (a), (b), (c) and (e). In particular, $\mathfrak{l}_b$ is stable under the adjoint action of Deligne's map u. So $(\mathfrak{l}_S, u(-1), u(\mathrm{i}))$ is a Hermitian symmetric Lie algebra.

Remark 4.8. Everything we have said in this section makes sense for a general subset $S \subseteq \{1, \ldots, r\}$, as in [AMSRT10]. Our case just corresponds to $S = \{1, \ldots, b\}$.

5. Standard boundary components

We keep the notations of the previous section. In particular,

$$\begin{aligned} \mathfrak{l}_b &= \sum_{\alpha \in {}_{\mathbb{R}}\Phi, \alpha = \sum_{i>b} a_i \alpha_i} \mathfrak{g}^{\alpha} + [\mathfrak{g}^{\alpha}, \mathfrak{g}^{-\alpha}] = \sum_{\alpha \in {}_{\mathbb{R}}\Phi, \alpha = \sum_{i>b} a_i \beta_i} \mathfrak{g}^{\alpha} + [\mathfrak{g}^{\alpha}, \mathfrak{g}^{-\alpha}]\,, \\ \mathfrak{l}'_b &= \sum_{\alpha \in {}_{\mathbb{R}}\Phi, \alpha = \sum_{i<b} a_i \alpha_i} \mathfrak{g}^{\alpha} + [\mathfrak{g}^{\alpha}, \mathfrak{g}^{-\alpha}]\,. \end{aligned} \tag{5.1}$$

We will regard $D = G/K$ as a bounded domain in $\mathfrak{p}_+$ using Harish-Chandra embedding. The goal of this section is to study some components of ∂D.

Write L_b (resp. L'_b) for the subgroup of $Z_G(A_b)$ with Lie algebra $\mathfrak{l}_b$ (resp. $\mathfrak{l}'_b$).

Define

$$D_b := L_b/(L_b \cap K)\,.$$

Proposition 5.1. *The set D_b is a bounded symmetric domain containing in D.*

Proof. We already know that D_b is a Hermitian symmetric space. It admits a natural embedding into D induced by $L_b \to G$. It follows that D_b does not have any compact factors. □

Proposition 5.2. *$\mathfrak{l}_b$ does not have compact factors.*

Proof. I do not understand the arguments in [AMSRT10, Page 124]. □

Proposition 5.3. *L_b commutes with $\prod_{i=1}^b \mathrm{SL}(2, \mathbb{R}) \subseteq G$ modulo $Z(G)$.*

Proof. This is clear by considering Lie algebras. □

We find natural embeddings

$$\begin{tikzcd} \Delta^b \times D_b \arrow[r, "f_1"] \arrow[d, hook] & D \arrow[d, hook] \\ \mathbb{C}^b \times \mathfrak{p}_{+,b} \arrow[r, "f_2"] \arrow[d, hook] & \mathfrak{p}_+ \arrow[d, hook] \\ (\mathbb{P}^1)^b \times \check{D}_b \arrow[r, "f_3"] & \check{D} \end{tikzcd}\,.$$

Here $\mathfrak{p}_{+,b} = \mathfrak{p}_+ \cap \mathfrak{l}_{b,\mathbb{C}}$. The map f_2 is just the inclusion of

$$\bigoplus_{i=1}^{b} \mathbb{C}X_i \oplus \mathfrak{p}_{+,S} \hookrightarrow \mathfrak{p}_+ \,.$$

Now we introduce the standard boundary components

$$F_b := f_2\left((1,\dots,1) \times D_b\right) \subseteq \partial D \subseteq \mathfrak{p}_+ \,.$$

Similarly, we write

$$\check{F}_b := f_3\left((1,\dots,1) \times \check{D}_b\right) \subseteq \check{D} \,.$$

We also regard D_b (resp. $\check{D}_b$) as a subset of D (resp. $\check{D}$).

Let us recall the notion of Harish-Chandra map:

thm:HC **Theorem 5.4.** *There is a morphism* $\mathrm{HC} : U_1 \times \mathrm{SL}(2,\mathbb{R})^r \to G$ *such that*

(1) $\varphi(z, h^{\mathrm{SL}}(z), \dots, h^{\mathrm{SL}}(z)) = u(z)^2$.

(2) The map

$$\mathrm{dHC} : \mathbb{R} \oplus \bigoplus_{i=1}^{r} \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g} \,.$$

restricted to the second component is given by

$$\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{bmatrix}, \dots, \begin{bmatrix} a_r & b_r \\ c_r & -a_r \end{bmatrix}\right) \mapsto \sum_{i=1}^{s} a_i x_i - \frac{b+c}{2} y_i + \frac{b-c}{2} \mathrm{i} h_i \,.$$

(3) The restriction of HC *to* $\mathrm{SL}(2,\mathbb{R})^r \to G$ *is algebraic.*

Recall that $u : U_1 \to G$ is the Deligne map and

$$h^{\mathrm{SL}}(e^{\mathrm{i}\theta}) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \,.$$

It will be more convenient to notice that u^2 in fact comes from an algebraic homomorphism $h_o : \mathbb{S} \to \mathcal{G}$, where $\mathcal{S}$ is the Deligne torus. More generally, for each $x \in D$, we can write $h_x : \mathbb{S} \to \mathcal{G}$ for the unique morphism fixing x and $h_x(z)$ acts on $T_x D$ as multiplication by z^2.

In our setting, we introduce version of Harish-Chandra map with respect to a standard boundary component now. We set

$$\mathrm{HC}_b : U_1 \times \mathrm{SL}(2,\mathbb{R}) \to G \,, \quad (z,x) \mapsto \mathrm{HC}(z; x, \dots, x, h^{\mathrm{SL}}(z), \dots, h^{\mathrm{SL}}(z)) \,, \tag{5.2}$$

where x occurs b-times.

From the lecture of Yuanyang, we know that HC_b induces symmetric embeddings, uniquely determined by the condition that i is mapped to the base point $o \in D$,

$$\begin{tikzcd} \mathbb{H} \arrow[r, "f_b"] \arrow[d, hook] & D \arrow[d, hook] \\ \mathbb{P}^1 \arrow[r, "f_b"] & \check{D} \end{tikzcd} .$$

Proposition 5.5. *The image of* ∞ *under* f_b *is* $\sum_{i=1}^{b} X_i \in \mathfrak{p}_+$.

In terms of the Cayley transform we introduce below,

$$\sum_{i=1}^{b} X_i = c_b(o) \,.$$

Proof. Observe that

$$\infty = \lim_{t\to\infty} \exp t \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \cdot \mathrm{i} \,.$$

So the image of ∞ is given by

$$\lim_{t\to\infty} \exp t \sum_{i=1}^{b} x_i \cdot o = \sum_{i=1}^{b} X_i$$

from the explicit formula in Part I. $\square$

We have shown that $f_b(\infty) \in F_b$.
Similarly,

Proposition 5.6. *The image of* 0 *under* f_b *is* $-\sum_{i=1}^{b} X_i \in \mathfrak{p}_+$.

Next we consider the partial Cayley transform:

$$c_b := \mathrm{HC}_b\left(1; \frac{1}{1-\mathrm{i}}\begin{bmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{bmatrix}\right) \in G_{\mathbb{C}}\,.$$

The mysterious matrix can be explained by the following simple computation: Write

$$\left(\frac{1}{1-\mathrm{i}}\begin{bmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{bmatrix}\right)^{-1} \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} \left(\frac{1}{1-\mathrm{i}}\begin{bmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{bmatrix}\right) = h^{\mathrm{SL}}(z) \tag{5.3}$$

for $z \in U_1$. So the explicit matrix is nothing but a matrix diagonalizing the $h^{\mathrm{SL}}(z)$'s at the same time.

lma:cbDF **Lemma 5.7.** *We have*

$$c_b(D_b) = F_b\,, \quad c_b(\check{D}_b) = \check{F}_b\,.$$

Here D_b *is identified with* $L_b \cdot o$.

Proof. We only have to handle the components $i \le b$. Take any $\ell \in L_b$, then

$$\ell \cdot o = \sum_{i>b} a_i X_i\,.$$

So we are reduced to compute the $P_+ K_{\mathbb{C}} P_-$ decomposition of c_b. This follows from an explicit computation. To be added. $\square$

We define a cocharacter $w_b : \mathbb{G}_m \to \mathcal{G}$ by

$$w_b(t) = \mathrm{HC}_b\left(1, \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}\right)\,.$$

prop:dw1 **Proposition 5.8.** w_b *is a well-defined cocharacter of* $\mathcal{G}$. *Moreover,* $\mathrm{d}w_b : \mathbb{R} \to \mathfrak{g}$ *is given by*

{eq:wb}
$$\mathrm{d}w_b(1) = \sum_{i=1}^{b} x_i\,. \tag{5.4}$$

Proof. That w_b defines an algebraic cocharacter follows from Theorem 5.4 (from the part of theorem which is not written down in [AMSRT10]!). The differential of w_b is computed using Theorem 5.4. $\square$

Comparing (5.4) with (4.1), we find that $w_b \in X_*(\mathcal{G})_{\mathbb{R}}$ lies in $\mathfrak{a}_b$.
We have already defined and studied the parabolic subgroup $\mathcal{P}_b = \mathcal{P}_{\mathcal{G}}(w_b)$. Its Lie algebra is computed in Proposition 4.5.

6. Normalizers and centralizers of standard boundary components

We use the same notations as above. But D is a general bounded symmetric domain, not necessarily irreducible.

6.1. **Normalizers.** Let $b = 1, \dots, r$ and F_b as before denotes the image of
$$(1, \dots, 1) \times D_b \subseteq \mathbb{C}^b \times \mathfrak{p}_{+,b}$$
in $\mathfrak{p}_+$. It is a standard boundary component of D.

Definition 6.1. The *normalizer* of F_b is the stabilizer $\mathcal{N}(F_b)$ of F_b in $\mathcal{G}$. In particular, it is an $\mathbb{R}$-algebraic subgroup of $\mathcal{G}$.

We want to study the structure of $\mathcal{N}(F_b)$ in this section.

Theorem 6.2. *We have*
$$\mathcal{P}_b = \mathcal{N}(F_b)\,.$$

It follows that $\mathcal{N}(F_b)$ is in fact connected, which is not *a priori* obvious.

Before the proof, recall that c_b induces bijections Lemma 5.7:
$$D_b \xrightarrow{\sim} F_b\,, \quad \check{D}_b \xrightarrow{\sim} \check{F}_b\,.$$
In particular, $c_b(o) \in F_b$.

Proof. We may assume that D is irreducible.

Step 1. We show that
$$\mathcal{P}_b \subseteq \mathcal{N}(F_b)\,.$$

It suffices to verify on $\mathbb{R}$-points. Take $g \in P_b$. From the lecture of Zhixiang, we know that gF_b is a boundary component. Two boundary components are either identical or disjoint. So it suffices to show that $g \cdot c_b(o) \in c_b(\check{D}_b) = \check{F}_b$, as $\check{F}_b \cap \bar{D} = c_b(\bar{D}_b)$ and hence $gF_b \subseteq \bar{F}_b$.

We are reduced to show that $c_b^{-1} g c_b(o) \in \check{D}_b$ or equivalently
$$c_b^{-1} g c_b(o) \in L_{b,\mathbb{C}} \cdot K_{\mathbb{C}} \cdot P_-\,. \tag{6.1}$$
The right-hand side is nothing but the stabilizer of $\check{D}_b$ in $G_{\mathbb{C}}$.

We will in fact prove more generally the following holds
$$c_b^{-1} \mathcal{P}_b(\mathbb{C}) c_b \subseteq L_{b,\mathbb{C}} \cdot K_{\mathbb{C}} \cdot P_-\,. \tag{6.2}$$
Observe that $c_b^{-1} \mathcal{P}_b(\mathbb{C}) c_b$ are the $\mathbb{C}$-points of $\mathcal{P}_{\mathcal{G}_{\mathbb{C}}}(w_b')$, where $w_b' = c_b^{-1} w_b c_b \in X_*(\mathcal{G})_{\mathbb{C}}$. From this description, we know that $\mathcal{P}_{\mathcal{G}_{\mathbb{C}}}(w_b')$ is parabolic and hence $c_b^{-1} \mathcal{P}_b(\mathbb{C}) c_b$ is connected (in complex topology as well as in Zariski topology). In particular it suffices to prove the corresponding result for Lie algebras of both sides of (6.2). But we know how to compute these Lie algebras (3.2), the remaining of the argument is just a long and tedious computation. We refer to [AMSRT10] Page 131 for the details.

Step 2. We already know that $\mathcal{P}_b$ is a maximal parabolic subgroup, it remains to show that $\mathcal{N}(F_b)$ is not equal to $\mathcal{G}$, but this is quite obvious, for example c_b^{-1} does not stabilize F_b. † □

Let us present a different point of view. Consider again our cocharacter $w_b : \mathbb{G}_m \to \mathcal{G}$. It induces a filtration $W_\bullet$ on $\mathfrak{g}$. Moreover, the Deligne's map $h_o : \mathbb{S} \to \mathcal{G}$ defines a decreasing filtration $\mathcal{F}^\bullet$ on $\mathfrak{g}$.

Theorem 6.3. $(\mathfrak{g}, W_\bullet, \mathcal{F}^\bullet)$ *is a* $\mathbb{R}$*-MHS.*

This follows from a more general theorem of Deligne. From the general theory of Caylay filtrations, $W_0\mathcal{G}$ can be defined and shown to be equal to $\mathcal{N}(F_b)$.

As w_b splits the weight filtration, we can write
$$\mathfrak{g} = \mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1} \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^1 \oplus \mathfrak{g}^2$$
with $\mathfrak{g}^i = \mathrm{Gr}^W_\bullet\,\mathfrak{g}$.

From the general theory (3.1), we know that
$$\mathcal{N}(F_b) = \mathcal{Z}(w_b) \ltimes \mathcal{W}(F_b)\,, \tag{6.3}$$

†The arguments in [AMSRT10] Page 132 are unnecessary, given the fact that all parabolic subgroups are connected.

where

$$\mathcal{W}(F_b) = \mathcal{U}_{\mathcal{G}}(w_b) = \mathcal{R}_u(\mathcal{N}(F_b))\,. \tag{6.4}$$

This can be easily rewritten using the general theory as

$$W_0\mathcal{G} = Z(w_b) \ltimes W_{-1}\mathcal{G}\,.$$

We observe that the Lie algebra of $Z(w_b)$ is nothing but $\mathfrak{g}^0$ by (3.2).

We want to have some further information about the decomposition (6.3).

The unipotent part is easy: observe that $W_{-3}G$ is in fact trivial. We let $\mathcal{U}(F_b) = W_{-2}\mathcal{G}$. Then $\mathcal{U}(F_b)$ is just the center of $\mathcal{W}(F_b)$ and

$$\mathcal{V}(F_b) := \mathcal{W}(F_b)/\mathcal{U}(F_b) = \mathrm{Gr}^W_{-1} G$$

is commutative. We have a short exact sequence

$$0 \to \mathcal{U}(F_b) \to \mathcal{W}(F_b) \to \mathcal{V}(F_b) \to 0\,.$$

Both $\mathcal{U}(F_b)$ and $\mathcal{V}(F_b)$ are commutative and unipotent, so they are just direct sums of $\mathbb{G}_a$ [Mil17, Proposition 14.32]. So we can more or less say that the structure of $\mathcal{W}(F_b)$ is well-understood. Finally, observe that the Lie algebra of $\mathcal{W}(F_b)$ is $\mathfrak{g}^{-2} \oplus \mathfrak{g}^{-1}$.

Next we handle the Levi part $\mathcal{Z}(w_b) = \mathfrak{g}^0$. We observe that Let $\mathfrak{g}_{\ell,b}$ be the orthogonal complement of $\mathfrak{l}_b$ in $\mathfrak{g}^0 = \mathfrak{z}(w_b)$. A direct computation shows that

$$\mathfrak{g}_{\ell,b} = \mathfrak{z}_{\mathfrak{g}}(\mathfrak{a}_b) \oplus \bigoplus_{\alpha = \pm\frac{1}{2}(\beta_i - \beta_j), 1 \leq i < j \leq b} \mathfrak{g}^{\alpha}\,.$$

Using our earlier notations,

$$\mathfrak{g}_{\ell,b} = \mathfrak{m}_b \oplus \mathfrak{a}_b \oplus \mathfrak{l}'_b\,.$$

In fact

$$\mathfrak{g}_{\ell,b} = [\mathfrak{g}^2, \mathfrak{g}^{-2}]\,.$$

Then we have

$$\mathfrak{g}^0 = \mathfrak{l}_b \oplus \mathfrak{g}_{\ell,b}\,.$$

This is the same as (4.7).

We can also lift the decomposition to an isogeny

$$L_b \times \mathcal{G}_\ell(F_b) \to \mathcal{Z}(w_b)\,.$$

6.2. **Centralizers.** Next we want to define the centralizer of a boundary component:

Definition 6.4. The *centralizer* of F_b is the unique $\mathbb{R}$-algebraic subgroup $\mathcal{Z}(F_b)$ of $\mathcal{G}$ such that

$$\mathcal{Z}(F_b)(\mathbb{C}) = \{g \in \mathcal{G}(\mathbb{C}) : gx = x \forall x \in F_b\}\,.$$ [‡]

Remark 6.5. The group $\mathcal{Z}(F_b)$ is not necessarily connected. In general, $\mathcal{Z}(F_b)$ is not the same as $\mathcal{Z}(w_b)$.

Theorem 6.6. *We have*

$$\mathcal{Z}(F_b)^0 = \mathcal{G}_\ell(F_b) \ltimes \mathcal{W}(F_b)\,.$$

Proof. We first check that $G_{\ell,b}$ acts identically on F_b. By definition, $G_{\ell,b}$ and L_b are commuting subgroups of $\mathcal{Z}(w_b)$, while L_b acts transitively on F_b by the construction of F_b. It follows that we only need to verify $g \in G_{\ell,b}(\mathbb{C})$ acts trivially on a single point. We take our favorite special point $c_b(o) \in F_b$. Then we need to show that $c_b^{-1} g c_b$ fixes o. Namely,

$$\mathrm{Ad}\, c_b(\mathfrak{g}_{\ell,b}) \subseteq \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_-\,.$$

For this, one just needs a direct computation.

[‡]The definition in [AMSRT10] Page 134 does not seem to be correct, as they require only conditions for $\mathbb{R}$-points.

Next we check that $\mathcal{W}(F_b)$ acts trivially on F_b. As $\mathcal{F}(F_b)$ is normal in $\mathcal{N}(F_b)$, it is in particular normalized by L_b as well. So again, we only need to show that any $g \in \mathcal{W}(F_b)(\mathbb{C})$ fixes $c_b(o)$. In terms of Lie algebras:

$$\operatorname{Ad} c_b(\mathfrak{w}(F_b)) \subseteq \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_- \,.$$

This again follows from a direct computation.

We have shown that

$$\mathcal{Z}(F_b)^0 \supseteq \mathcal{G}_\ell(F_b) \ltimes \mathcal{W}(F_b)\,. \tag{6.5}$$

Next observe that $\mathcal{N}(F_b)/(\mathcal{G}_\ell(F_b) \ltimes \mathcal{W}(F_b))$ is nothing but $L_b/(L_b \cap \mathcal{G}_\ell(F_b))$, the latter is just L_b divided by its center, which acts faithfully on F_b. This readily implies that the equality in (6.5) holds. □

Another important aspect of $\mathcal{G}_\ell(F_b)$ is the following:

Proposition 6.7. *$\mathcal{G}_\ell(F_b)$ fixes $f_b(0)$.*

Proof. Let $s_o : \check{D} \to \check{D}$ be the geodesic symmetry with respect to o. Let $g \in \mathcal{G}_\ell(F_b)(\mathbb{C})$, then

$$g f_b(0) = g s_o(f_b(\infty)) = s_o\sigma(g)(f_b(\infty)) = s_o(f_b(\infty)) = f_b(0)\,.$$

Here σ denotes the Cartan involution of $\mathcal{G}_{\mathbb{C}}$. □

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