ON LIU MORPHISMS IN NON-ARCHIMEDEAN GEOMETRY

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Abstract. We define Liu morphisms and quasi-Liu morphisms between Berkovich analytic spaces. We show that Liu morphisms and quasi-Liu morphisms behave as affine morphisms and quasi-affine morphisms of schemes in many aspects.

Contents

1. Introduction

1.1. **Motivation.** In classical algebraic geometry, the theories of affine morphisms and quasi-affine morphisms play a prominent role. In the non-Archimedean world, it is highly desirable to have analogous results as well. However, there are two principal difficulties in the non-Archimedean setting:

(1) First of all, there is no satisfactory theory of quasi-coherent sheaves in non-Archimedean geometry. There is indeed an *ad hoc* notion of quasi-coherent sheaves in rigid geometry defined by Conrad in [Con06]: A quasi-coherent sheaf is a sheaf of modules which can be expressed as a filtered colimit of coherent sheaves locally. However, Conrad's notion of quasi-coherent sheaves does not behave as expected: On an affinoid space, the higher cohomologies of a quasi-coherent sheaf do not vanish in general. This makes it hard to handle affine morphisms in terms of quasi-coherent sheaves of algebras. The same problem persists in Berkovich geometry.
(2) Secondly, a more severe problem was proposed by Liu [Liu88], [Liu90]. It is shown that there is a quasi-compact, separated non-affinoid rigid space X, a morphism $f : X \to Y$ to an affinoid space Y, an admissible affinoid covering $\{U_i\}$ of Y such that $f^{-1}U_i$ is affinoid for each i. See [Liu90, Proposition 3.3 and Section 5]. This means that the property that the inverse image of an affinoid domain is affinoid is *not* G-local.

Recall that in classical algebraic geometry, we have the celebrated *Serre's criterion* ([EGA II, Théorème 5.2.1]): Affine schemes can be characterized by cohomological triviality among quasi-compact separated schemes. Similarly, in non-Archimedean setting, we replace the usual local notion of affinoid spaces by cohomologically trivial spaces. Such spaces are studied by Maculan–Poineau in [MP21] under the name of *Liu spaces*, we follow their terminology.

Definition 1.1 (c.f. Definition 3.1)**.** Let k be a complete non-Archimedean valued field. A quasi-compact, separated k-analytic space X (in the sense of Berkovich) is said to be *Liu* if for any analytic extension k'/k, any coherent sheaf $\mathcal{F}$ on $X_{k'}$ is acyclic.

Date: August 16, 2025.

On the morphism level, we define a *Liu morphism* as a morphism under which the inverse image of a Liu domain is a Liu space, see Definition 4.1. Similarly, we have a notion of quasi-Liu morphisms analogous to the classical notion of quasi-affine morphisms:

Definition 1.2 (c.f. Definition 5.2)**.** Let $f : X \to Y$ be a morphism of k-analytic spaces. We say f is *quasi-Liu* if for any Liu domain Z in Y, $f^{-1}Z$ can be embedded in a Liu k-analytic space as a compact analytic domain and $H^0(f^{-1}Z, \mathcal{O}_X)$ is a Liu k-algebra (Definition 3.4).

Similar to the situation in classical algebraic geometry, we prove a cohomological criterion of Liu morphisms when Y is Liu in Theorem 4.4.

Unfortunately, as pointed out by Marco Maculan, contrary to the assertion in the previous versions of this paper, the notion of Liu morphisms is not G-local on the target, see an example due to Scholze–Weinstein in Example 4.1.

As for (1), due to the progress made by Ben-Bassat–Kremnizer in [BBK17], it is by far clear that the natural notion on a non-Archimedean analytic space is not that of the quasi-coherent sheaves, but the derived category of quasi-coherent sheaves instead. However, as we will see, in the special case of sheaves of Liu algebras studied below, the derived notion reduces to a *bona fide* notion of quasi-coherence at the non-derived level. In particular, on a separated space, there is a global notion of quasi-coherent sheaves of Liu algebras, see Definition 4.2.

1.2. **Main results.** We fix a complete non-Archimedean valued field k. We allow the valuation on k to be trivial. We work in the framework of Berkovich spaces as in [Ber93].

The main result says that Liu morphisms and quasi-coherent sheaves of Liu k-algebras are essentially equivalent:

Theorem 1.1 (=Corollary 4.7)**.** *Let X be a separated k-analytic space. Then the functor*

$$\underline{\mathrm{Sp}}_X : \mathcal{L}\mathrm{iu}\mathcal{A}\mathrm{lg}_{X,k}^{\mathrm{QCoh}} \to \mathcal{L}\mathrm{iu}_{\to X,k}$$

is an anti-equivalence of categories.

Here $\mathcal{L}\mathrm{iu}\mathcal{A}\mathrm{lg}_{X,k}^{\mathrm{QCoh}}$ is the category of quasi-coherent sheaves of Liu k-algebras on X, $\mathcal{L}\mathrm{iu}_{\to X,k}$ is the category of Liu morphisms $Y \to X$. The functor $\underline{\mathrm{Sp}}_X$ is the relative spectrum functor defined in Definition 4.3. This result is analogous to the classical result on affine morphisms and quasi-coherent sheaves of algebras ([EGA II, Proposition 1.2.7, Proposition 1.3.1]).

1.3. **Structure of the paper.** In Section 2, we recall some basic results about Berkovich analytic spaces and the language developed by Ben-Bassat and Kremnizer ([BBK17]). Due to the lack of references, we also prove a representability theorem (Theorem 2.1) about presheaves on the category of analytic spaces.

In Section 3, we recall the basic theory of Liu spaces and Liu algebras. We prove that Liu algebras behave very similar to affinoid algebras in many aspects.

In Section 4, we introduce Liu morphisms and study their relation to quasi-coherent sheaves of Liu algebras.

In Section 5, we introduce and study quasi-Liu morphisms.

In Section 6, we give a list of unsolved problems related to this work.

We collect results from [BBK17] in Appendix A.

1.4. **Conventions.** Let k be a complete non-Archimedean valued field. An *analytic extension* of k is a complete non-Archimedean valued field k' containing k such that the restriction of the valuation on k' to k coincides with the given valuation on k. We denote the spectrum of a Banach algebra A by $\mathrm{Sp}\, A$ instead of the more common notation $\mathcal{M}(A)$.

1.5. **Acknowledgments.** I would like to thank Yanbo Fang for discussions, Jérôme Poineau for comments on the draft and Michael Temkin for answering questions about locally affinoid algebras. I am indebted to the anonymous referee for many valuable suggestions and especially for pointing out several mistakes in the original version of the manuscript. I would like to thank Marco Maculan for pointing out a mistake in the original Theorem 4.4.

2. Preliminaries

Let k be a complete non-Archimedean valued field.

2.1. **Analytic spaces.** In this paper, by a k-analytic space, we mean a k-analytic space in the sense of [Ber93]. The category of k-analytic spaces is denoted by $\mathcal{A}\mathrm{n}_k$. For each k-analytic space X, we endow X with the G-topology as in [Ber93]. The corresponding site is still denoted by X. There is a natural sheaf of rings $\mathcal{O}_{X_G}$ making X a ringed site. We always omit the subindex G and write $\mathcal{O}_X$ instead. The category of coherent sheaves on X is denoted by $\mathcal{C}\mathrm{oh}_X$.

Strict k-analytic spaces are defined as in [Ber93]. Recall that by a celebrated result of Temkin [Tem04], strict k-analytic spaces form a full subcategory of the category of k-analytic spaces if k is non-trivially valued. The category of k-affinoid spaces is denoted by $\mathcal{A}\mathrm{ff}_k$, see [Ber12]. The category of k-affinoid algebras is denoted by $\mathcal{A}\mathrm{ff}\mathcal{A}\mathrm{lg}_k$. There is an equivalence between $\mathcal{A}\mathrm{ff}_k$ and $\mathcal{A}\mathrm{ff}\mathcal{A}\mathrm{lg}_k$, given by the functor of global sections $X \mapsto H^0(X, \mathcal{O}_X)$ and the functor of Berkovich spectrum $A \mapsto \operatorname{Sp} A$.

2.2. **A representability theorem.** The following result is analogous to [EGA I-new, Proposition 4.5.4].

Theorem 2.1. *Let F be a presheaf on $\mathcal{A}\mathrm{n}_k$. Assume that*

(1) F satisfies the sheaf property for the G-topology, namely, for any k-analytic space X, any G-covering $\{U_i\}$ of X, $F(X)$ is the equalizer of

$$\prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \cap U_j)\,.$$

(2) There is a family $\{F_i\}_i$ of subfunctors of F such that
   - *(a) Each F_i is representable by a k-analytic space X_i.*
   - *(b) Each $F_i \to F$ is representable by a* closed *(resp.* open*) analytic domain. In particular, after base change to X_j, $F_i \to F$ is represented by a closed (resp. open) analytic domain U_{ji}. In the closed case, we assume furthermore that for each i, the collection of j such that $U_{ij} \neq \emptyset$ is finite.*
   - *(c) The collection F_i covers F.*

Then F is representable.

Proof. Let $\xi_i \in F_i(X_i)$ be the universal family of the presheaf F_i. By assumption a morphism of k-analytic spaces $T \to X_i$ factors through U_{ij} iff $\xi_i|_T \in F_j(T)$. In particular, $\xi_i|_{U_{ij}} \in F_j(U_{ij})$. So we get a morphism $f_{ij} : U_{ij} \to X_j$ such that $f_{ij}^*\xi_j = \xi_i|_{U_{ij}}$. By definition of U_{ji}, we know that f_{ij} factors through U_{ji}. Now observe that $(f_{ij} \circ f_{ji})^*\xi_j = f_{ji}^*\xi_i = \xi_j$, we conclude that $f_{ij} \circ f_{ji} = \mathrm{id}_{U_{ji}}$. In particular, all f_{ij} are isomorphisms. It is formal to see that the glueing conditions are satisfied by the f_{ij}'s, hence we can glue the X_i's together to get a k-analytic space X by [Ber93, Proposition 1.3.3]. It is formal to check that X together with the glueing ξ of ξ_i represents F. We refer to [Stacks, Tag 01JJ] for the omitted details. □

2.3. **Polyradii.**

Definition 2.1. A *polyradius* is an element $r \in \mathbb{R}^n_{>0}$ for some $n \in \mathbb{N}$. A polyradius r is *k-free* if the components of r are linearly independent as elements in the $\mathbb{Q}$-vector space $\mathbb{Q} \otimes_{\mathbb{Z}} (\mathbb{R}_{>0}/|k^*|)$.

For any k-polyradius $r \in \mathbb{R}^n_{>0}$, define k_r as the k-affinoid algebra of formal series

$$\left\{ \sum_{\alpha\in\mathbb{Z}^n} a_\alpha T^\alpha \in k[[T_1, \dots, T_n]] \mid a_\alpha \in k, |a_\alpha| r^\alpha \to 0 \text{ when } |\alpha| \to \infty \right\}$$

endowed with the multiplicative norm $\sum_{\alpha\in\mathbb{Z}^n} a_\alpha T^\alpha \mapsto \max_{\alpha\in\mathbb{Z}^n} a_\alpha T^\alpha$.

When r is k-free, k_r is a field.

For a given k-free polyradius r, a given Banach k-algebra A, for any Banach A-module M, we write $A_r = A\hat{\otimes}_k k_r$, $M_r = M\hat{\otimes}_k k_r$. Note that A_r is a Banach k_r-algebra and M_r is a Banach A_r-module. Similarly, given any k-analytic space, write $X_r := X \times_{\operatorname{Sp} k} \operatorname{Sp} k_r$.

2.4. **The category of Banach modules.** We briefly summarize a few results in [BBK17]. For the basic theory of quasi-Abelian categories, see [Sch99].

Let $\mathcal{B}\mathrm{an}_k$ be the category of Banach k-modules, where morphisms are bounded homomorphisms. Recall that $\mathcal{B}\mathrm{an}_k$ is a closed symmetric monoidal quasi-Abelian category with all finite limits and finite colimits, where the $\otimes$ operator is given by the completed tensor product $\hat{\otimes}$. Moreover, finite products and finite coproducts coincide. The category $\mathcal{B}\mathrm{an}_k$ has enough projectives. All projective objects in $\mathcal{B}\mathrm{an}_k$ are flat in the sense of [BBB16]. We have derived categories $\mathrm{D}^*(\mathcal{B}\mathrm{an}_k)$, where $*$ means $+$, $-$, b or empty. Let $\mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}_k$ be the category of Banach k-algebras, which is also the category of algebras in the symmetric monoidal category $\mathcal{B}\mathrm{an}_k$ in the abstract sense. Let $A \in \mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}_k$ be a Banach k-algebra. Let $\mathcal{B}\mathrm{an}\mathcal{M}\mathrm{od}_A$ be the category of Banach A-modules, which is also the category of A-modules in the symmetric monoidal category $\mathcal{B}\mathrm{an}_k$ in the abstract sense. Recall that $\mathcal{B}\mathrm{an}\mathcal{M}\mathrm{od}_A$ is also a closed symmetric monoidal quasi-Abelian category with all finite limits and finite colimits, where the $\otimes$ operator is also given by $\hat{\otimes}$. We write $\mathrm{D}^*(A) = \mathrm{D}^*(\mathcal{B}\mathrm{an}\mathcal{M}\mathrm{od}_A)$.

Definition 2.2. Let $f : A \to B$ be a morphism in $\mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}_k$. Let $M \in \mathcal{B}\mathrm{an}\mathcal{M}\mathrm{od}_A$. We say that M is *transversal* to f if the natural morphism

$$M\hat{\otimes}^{\mathbb{L}}_A B \to M\hat{\otimes}_A B$$

in $\mathrm{D}^-(A)$ is an isomorphism.

Proposition 2.2 ([Ber12, Proposition 2.1.2])**.** *For any k-free polyradius, the Banach k-module k_r is flat in $\mathcal{B}\mathrm{an}_k$: for any admissible exact short sequence $0 \to E \to F \to G \to 0$ in $\mathcal{B}\mathrm{an}_k$, the following sequence is also admissible and exact:*

$$0 \to E_r \to F_r \to G_r \to 0\,.$$

3. Liu spaces and Liu algebras

Let k be a complete non-Archimedean valued field.

3.1. **Liu spaces.** In this section, we recall the basic theory of Liu k-analytic spaces following [MP21] and [Liu90].

Definition 3.1 ([MP21, Definition 1.9])**.** A k-analytic space X is called *Liu* if

(1) X is quasi-compact, separated.
(2) X is holomorphically separable: for any $x, y \in X$, $x \neq y$, there is $f \in H^0(X, \mathcal{O}_X)$ such that $|f(x)| \neq |f(y)|$.
(3) $\mathcal{O}_X$ is universally acyclic: for any analytic extension k'/k, $H^i(X_{k'}, \mathcal{O}_{X,k'}) = 0$ for any $i > 0$.

A morphism of Liu k-analytic spaces is a morphism of the underlying k-analytic spaces. We denote the category of Liu k-analytic spaces by $\mathcal{L}\mathrm{iu}_k$.

Example 3.1. *A k-affinoid space is a Liu k-analytic space. But the converse fails in general. We refer to [Liu90, Section 5] for details. In fact, the theory of non-Archimedean pinching in [Tem21] gives plenty of such examples.*

Definition 3.2. Let X be a k-analytic space. An analytic domain Z of X is called a *Liu domain* if Z is a Liu k-analytic space.

Definition 3.3. Let X be a k-analytic space. We say X is *cohomologically Stein* if for any coherent sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$,

$$H^i(X, \mathcal{F}) = 0\,, \quad i > 0\,.$$

We say X is *universally cohomologically Stein* if for any analytic extension k'/k, $X_{k'}$ is cohomologically Stein.

Theorem 3.1 ([MP21, Theorem 1.11], [Liu90, Théorème 2])**.** *Let X be a separated, quasi-compact k-analytic space. Then the following are equivalent:*

(1) X is Liu.
(2) X is universally cohomologically Stein.
(3) X is holomorphically separable and $\mathcal{O}_X$ is universally acyclic.

Moreover, if k is non-trivially valued and X is strict, then the conditions are equivalent to

(4) X is rig-holomorphically separable and $\mathcal{O}_X$ is acyclic.

Note that in (4), we only need acyclicity of $\mathcal{O}_X$ instead of universal acyclicity as explained in [MP21]. For the definition of rig-holomorphically separability, we refer to [MP21, Definition 1.5].

Theorem 3.2 ([MP21, Corollary 1.16])**.** *Let $f : Y \to X$ be a finite morphism of k-analytic spaces. Then*

(1) If X is Liu, then so is Y.
(2) If Y is Liu and f is surjective, then X is Liu.

Theorem 3.3 ([MP21, Corollary 1.15, Corollary 1.17])**.** *Let X be a k-analytic space. Then*

(1) For any analytic extension k'/k, $X_{k'}$ is Liu iff X is Liu.
(2) Assume that X is separated. Then X is Liu iff X^{red} is.
(3) Assume that X is separated. Then X is Liu iff each irreducible component of X is.

Proof. We only have to make the following remark to (1): X is separated iff $X_{k'}$ is. This follows from [CT21, Theorem 1.2]. □

Proposition 3.4. *Let $f : Y \to X$, $g : X' \to X$ be morphisms in $\mathcal{L}\mathrm{iu}_k$. Then $Y' := Y \times_X X' \in \mathcal{L}\mathrm{iu}_k$.*

Proof. We have the following Cartesian diagram

$$\begin{array}{ccc} Y' & \longrightarrow & Y \times X' \\ {\scriptstyle (f,g)}\downarrow & \square & \downarrow{\scriptstyle f\times g} \\ X & \xrightarrow{\Delta_X} & X \times X \end{array} .$$

As X is separated, Δ_X is a closed immersion, so is the morphism $Y' \to Y \times X'$. By Theorem 3.2, in order to show that Y' is Liu, it suffices to show that $Y \times X'$ is Liu. This follows from [MP21, Theorem A.6]. □

Corollary 3.5. *Let X be a separated k-analytic space. Let Y_1, Y_2 be Liu domains in X, then $Y_1 \cap Y_2$ is also a Liu domain.*

3.2. **Liu algebras.**

Definition 3.4. A *Liu k-algebra* is a Banach k-algebra A such that there is a Liu k-analytic space such that $A \cong H^0(X, \mathcal{O}_X)$, where the isomorphism is an isomorphism of Banach k-algebras. A Liu k-algebra is said to be *strict* if there is a strict Liu k-analytic space with $A \cong H^0(X, \mathcal{O}_X)$ in $\mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}_k$.

A morphism of Liu k-algebras is a bounded homomorphism of the underlying Banach k-algebras.

The category of Liu k-algebras is denoted by $\mathcal{L}\mathrm{iu}\mathcal{A}\mathrm{lg}_k$. It is a full subcategory of $\mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}_k$.

Proposition 3.6. *Let A be a Liu k-algebra. Then*

(1) A is Noetherian and all of its ideals are closed.
(2) Suppose that k is non-trivially valued and A is strict. For any maximal ideal $\mathfrak{m}$ of A, $A/\mathfrak{m}$ is finite dimensional as a vector space over k.
(3) We have

$$\bigcap_{\mathfrak{m}\in\mathrm{Max}(A)} \bigcap_{n=1}^{\infty} \mathfrak{m}^n = 0 .$$

Proof. (1) That A is noetherian follows from [MP21, Proposition 2.6(3), Remark 2.7]. When k is non-trivially valued, all ideals are closed by [BGR84, Proposition 3.7.2.2]. In general, this follows from a base field extension argument, see [Ber12, Proposition 2.1.3].

(2) By [Liu90, Proposition 1.3], there is a rigid point $x \in X$ such that $\mathfrak{m} = \mathfrak{m}_{\mathrm{Sp}\,A,x}$. Take a strictly affinoid domain $\mathrm{Sp}\,B$ of $\mathrm{Sp}\,A$ containing x. Then x is also rigid in $\mathrm{Sp}\,B$. It is well-known that $B/\mathfrak{m}_{\mathrm{Sp}\,B,x}$ is finite dimensional, hence so is $A/\mathfrak{m}$.

(3) Take an element $a \in A$ that lies in the intersection of all $\mathfrak{m}^n$ for any $\mathfrak{m} \in \mathrm{Max}\,A$, $n \geq 1$. Then By Krull's intersection theorem, for each $\mathfrak{m} \in \mathrm{Max}\,A$, there is an element $m \in \mathfrak{m}$ such that $(1-m)a = 0$. Thus the annihilator of a does not lie in any maximal ideal of A, hence $a = 0$. □

Corollary 3.7. *Let A be a Liu k-algebra. All k-algebra homomorphisms from a Banach k-algebra to A are bounded. In particular, the Liu k-algebra structure of A is uniquely determined by the underlying algebraic structure.*

Proof. When k is non-trivially valued and A is strict, this follows from Proposition 3.6 and [BGR84, Proposition 3.7.5.2].

In general, this follows from the change of base argument. □

Theorem 3.8 (Liu)**.** *The functor of global sections gives an anti-equivalence $\mathcal{L}\mathrm{iu}_k \to \mathcal{L}\mathrm{iu}\mathcal{A}\mathrm{lg}_k$. The inverse functor is denoted by $\operatorname{Sp} A$. Moreover, for any k-analytic space Y, any Liu k-analytic space X, the canonical map*

$$\operatorname{Hom}_{\mathcal{A}\mathrm{n}_k}(Y,X) \to \operatorname{Hom}_{\mathcal{A}\mathrm{lg}_k}(H^0(X,\mathcal{O}_X), H^0(Y,\mathcal{O}_Y))$$

is bijective.

Remark 3.1. The space $\operatorname{Sp} A$ as a topological space coincides with the spectrum in the sense of Berkovich [Ber12, Section 1.2]. See [MP21, Corollary 3.17] for example.

Proof. The latter statement is a formal consequence of the former.

When k is non-trivially valued, by [Liu90, Proposition 3.2] and [Ber93, Theorem 1.6.1], we know that the global section functor is an anti-equivalence from the category of strict Liu k-analytic spaces to the category of strict Liu k-algebras.

In general, let X, Y be Liu k-analytic spaces. Let $A = H^0(X,\mathcal{O}_X)$, $B = H^0(Y,\mathcal{O}_Y)$. Let $F : A \to B$ be a homomorphism of k-algebras. We want to construct a morphism $Y \to X$, whose induced map on global sections is given by F. We may assume that Y is affinoid. Take an analytic field extension k'/k, so that k' is non-trivially valued, $A_{k'}$ and $B_{k'}$ become strict Liu k-algebras. We may assume that $k' = k_r$ for some k-free polyradius. Then there is a unique morphism $g : Y_{k'} \to X_{k'}$ inducing $F_{k'}$. We claim that there is a unique morphism $f : Y \to X$ such that $g = f_{k'}$. Note that it is automatic that f induces F on global sections by [Ber12, Proposition 2.1.2].

By [MP21, Proposition 3.13], there is a k-affinoid space Z, a locally closed immersion $h : X \to Z$ such that there is a finite covering $Z_1, \dots, Z_m$ of Z by rational domains such that $h^{-1}(Z_i) \to Z_i$ is a Runge immersion for each i:

$$\begin{array}{ccccc} Y_{k'} & \xrightarrow{g} & X_{k'} & \xrightarrow{h_{k'}} & Z_{k'} \\ \downarrow & & \downarrow & & \downarrow \\ Y & \overset{f}{\dashrightarrow} & X & \xrightarrow{h} & Z \end{array} \qquad w : Y \to Z .$$

Now observe that the composition of maps on global sections

$$H^0(Z_{k'},\mathcal{O}_{Z_{k'}}) \to H^0(X_{k'},\mathcal{O}_{X_{k'}}) \to H^0(Y_{k'},\mathcal{O}_{Y_{k'}})$$

is the same as the base extension of the map of k-algebras

$$H^0(Z,\mathcal{O}_Z) \to A \xrightarrow{F} B .$$

Thus if we denote by $w : Y \to Z$ the morphism of k-analytic spaces corresponding to this latter map, we have $w_{k'} = h_{k'} \circ g$. Replacing Y by $w^{-1}(Z_i)$, X by $h^{-1}Z_i$ and Z by Z_i and applying [Ber93, Proposition 1.3.2] and (3.1), we may assume that $X \to Z$ is a Runge immersion. In particular X is affinoid. We can take f to be the morphism corresponding to F. Moreover, such f (such that $g = f_{k'}$) is clearly unique. We conclude. □

Lemma 3.9. *Let A be a Liu k-algebra. Let B, C be Liu k-algebras over A, then $B\hat{\otimes}_A C$ is Liu. In particular, for any k-free polyradius r, A_r is a Liu k-algebra.*

Proof. Let $Z = \operatorname{Sp} B \times_{\operatorname{Sp} A} \operatorname{Sp} C$. By Proposition 3.4, it suffices to prove

$$H^0(Z,\mathcal{O}_Z) = B\hat{\otimes}_A C . \tag{3.1}$$

Firstly, we consider the morphism

$$\Delta_{\operatorname{Sp} A} : \operatorname{Sp} A \to \operatorname{Sp} A \times \operatorname{Sp} A .$$

It is easy to see that this is a closed immersion, corresponding to the closed ideal J in $A\hat{\otimes}A$ generated by $1\otimes a - a\otimes 1$ for $a\in A$. Also by [PP, Corollary 3.30], we have

$$H^0(\operatorname{Sp} B\times \operatorname{Sp} C, \mathcal{O}_{\operatorname{Sp} B\times\operatorname{Sp} C}) = B\hat{\otimes}C\,.$$

Hence the closed immersion $Z\to \operatorname{Sp} B\times\operatorname{Sp} C$ corresponds to the closed ideal of $B\hat{\otimes}C$ generated by $J(B\hat{\otimes}C)$. In particular, (3.1) holds. □

Definition 3.5. Let A be a Liu k-algebra. A Banach A-module M is *finite* if there is an admissible epimorphism $A^n\to M$.

Let $\mathcal{M}\mathrm{od}^{\mathrm{fin}}(A)$ be the category of finite A-modules.

Proposition 3.10. *Let A be a Liu k-algebra. The forgetful functor from the category of finite Banach A-modules (with bounded A-algebra homomorphisms as morphisms) to $\mathcal{M}\mathrm{od}^{\mathrm{fin}}(A)$ is an equivalence.*

Proof. The functor is fully faithful. In fact, we prove more generally that for any finite Banach A-module M, any Banach A-module N, any A-linear map $F: M\to N$ is bounded. In fact, taking an admissible epimorphism $A^n\to M$, we may assume that $M=A^n$. In this case, the claim is clear.

The functor is essentially surjective. Take an A-linear epimorphism $\pi: A^n\to M$, then $\ker\pi$ is closed by Proposition 3.6 (1) and [BGR84, Proposition 3.7.2.2], so we can endow M with the residue Banach norm. □

Proposition 3.11. *Let A be a Liu k-algebra. Let r be a k-free polyradius. Let M be Banach A-module. Then M is a finite Banach A-module iff M_r is a finite Banach A_r-module.*

Proof. This follows *verbatim* from [Ber12, Proof of Proposition 2.1.11]. □

Theorem 3.12. *Let $X=\operatorname{Sp} A$ be a Liu k-analytic space. Let r be a k-free polyradius. Consider a descent datum (M_r,φ) of Banach modules over A_r. Then the descent datum is effective with respect to the natural morphism $\operatorname{Sp} A_r\to\operatorname{Sp} A$. Moreover, if M_r is finitely generated as A_r-module, then the descent M is finitely generated as A-module.*

Proof. The first part follows *verbatim* from [Day21, Proof of Proposition 3.3]. The second part follows from Proposition 3.11. □

3.3. Coherent sheaves on Liu k-analytic spaces.

Definition 3.6. Let $X=\operatorname{Sp} A$ be a Liu k-analytic space. Let M be a finite A-module. Then we define a sheaf $\widetilde{M}$ on X as the sheafification of the presheaf $\operatorname{Sp} B\mapsto M\otimes_A B$, where $\operatorname{Sp} B$ runs over the set of affinoid domains in X.

Proposition 3.13 ([MP21, Lemma 2.4])**.** *Let $X=\operatorname{Sp} A$ be a Liu k-analytic space. Let M be a finite A-module. Then $\widetilde{M}$ is a coherent sheaf on X. Moreover, for each affinoid domain $\operatorname{Sp} B$ in X,*

$$H^0(\operatorname{Sp} B,\widetilde{M}) = M\otimes_A B\,. \tag{3.2}$$

Now we recall the theory of coherent sheaves on Liu k-analytic spaces. The following is the analogue of Cartan's Theorem A.

Theorem 3.14 ([MP21, Proposition 2.1])**.** *Assume that k is non-trivially valued. Let X be a Liu k-analytic space. For each coherent sheaf $\mathcal{F}$ on X and each $x\in X$, $H^0(X,\mathcal{F})$ generates $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-module.*

In the rigid setting, Cartan's Theorem A and Theorem B are due to Kiehl [Kie67] and Tate [Tat71] respectively.

As explained in [Kie67], Theorem A and Theorem B together imply the following result:

Theorem 3.15. *Let $X=\operatorname{Sp} A$ be a Liu k-analytic space. Then the category of coherent sheaves on X is equivalent to the category of finite A-modules. The functors are given by $\mathcal{F}\mapsto H^0(X,\mathcal{F})$ and $M\mapsto\widetilde{M}$ respectively.*

Proof. This result was proved in [MP21, Proposition 2.6] under the assumption that k is non-trivially valued. When k is trivially valued, take a k-free polyradius r with at least one component. By [Day21, Théorème 3.13], the category of coherent sheaves on X is equivalent to the category of descent data of coherent sheaves on X_r with respect to $X_r\to X$. The latter category is equivalent to the category of descent data of finite A_r-modules with respect to $A \to A_r$, which is then equivalent to the category of finite A-modules by Theorem 3.12. It is easy to see that the composition of these functors is exactly the one given in the theorem. The functors in the proof are summarized in the following diagram:

$$\begin{array}{ccc} \mathcal{D}\mathrm{es}(\mathcal{C}\mathrm{oh}, X_r \to X) & \longrightarrow & \mathcal{D}\mathrm{es}(\mathcal{M}\mathrm{od}^{\mathrm{fin}}, A \to A_r) \\ \downarrow & & \downarrow \\ \mathcal{C}\mathrm{oh}(X) & \longrightarrow & \mathcal{M}\mathrm{od}^{\mathrm{fin}}(A) \end{array}.$$

□

In particular, Theorem 3.14 holds even when k is trivially valued.

3.4. Quasi-coherent sheaves on Liu spaces.

Definition 3.7. Let $f : A \to B$ be a morphism in $\mathcal{L}\mathrm{iu}\mathcal{A}\mathrm{lg}_k$. We say f is a *homotopy epimorphism* if the corresponding morphism $\mathrm{Sp}\, B \to \mathrm{Sp}\, A$ of Liu k-spaces identifies $\mathrm{Sp}\, B$ with a Liu domain in $\mathrm{Sp}\, A$.

Definition 3.8. Let A be a Liu k-algebra. A Banach A-module M is called *transversal* if M is transversal to all homotopy epimorphisms from A: for all homotopy epimorphism $A \to B$ to a Liu k-algebra B, the natural morphism

$$M \hat{\otimes}^{\mathbb{L}}_A B \to M \hat{\otimes}_A B$$

is an isomorphism.

The following result will be proved in Appendix A.

Theorem 3.16. *Let A be a Liu k-algebra. Let B, C be Liu k-algebras over A such that $\mathrm{Sp}\, C \to \mathrm{Sp}\, A$ is a Liu domain. Then the natural morphism*

$$C \hat{\otimes}^{\mathbb{L}}_A B \to C \hat{\otimes}_A B$$

is an isomorphism.

Definition 3.9. Let A be a Liu k-algebra. Let M be a transversal Banach A-module. Write $X = \mathrm{Sp}\, A$. We define a sheaf of $\mathcal{O}_X$-modules $\widetilde{M}$ as the sheafification of the presheaf

$$\mathrm{Sp}\, B \mapsto M \hat{\otimes}_A B$$

on X, where $\mathrm{Sp}\, B$ runs over the set of affinoid domains in X. We call $\widetilde{M}$ the sheaf associated to M.

An $\mathcal{O}_X$-module $\mathcal{M}$ is *quasi-coherent* if there is a transversal A-module M such that $\mathcal{M} = \widetilde{M}$.

Example 3.2. *Let X be a Liu k-analytic space. Then all coherent sheaves on X are quasi-coherent. See for example [MP21, Proof of Proposition 2.6(1)]. To be more precise, the same proof shows that for any Liu domain $\mathrm{Sp}\, B \to \mathrm{Sp}\, A = X$, B is a flat A-algebra. Let M be a finite A-module. Consider a presentation*

$$A^{\oplus S} \to A^{\oplus N} \to M \to 0 .$$

We have a commutative diagram with exact rows:

$$\begin{array}{ccccccc} A^{\oplus S} \hat{\otimes}^{\mathbb{L}}_A B & \longrightarrow & A^{\oplus N} \hat{\otimes}^{\mathbb{L}}_A B & \longrightarrow & M \hat{\otimes}^{\mathbb{L}}_A B & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ A^{\oplus S} \otimes_A B & \longrightarrow & A^{\oplus N} \otimes_A B & \longrightarrow & M \otimes_A B & \longrightarrow & 0 \end{array}.$$

In order to show that M is transversal, it suffices to show that A is, which is obvious.

Theorem 3.17 (Tate acyclicity theorem)**.** *Let $X = \mathrm{Sp}\, A$ be a Liu k-analytic space. Let $\mathrm{Sp}\, A_1, \dots, \mathrm{Sp}\, A_n$ be a finite G-covering of X by Liu domains. Let M be a transversal Banach A-module, then the following sequence is admissible and exact*

$$0 \to M \to \prod_{i_1} M \hat{\otimes}_A A_{i_1} \to \prod_{i_1 < i_2} M \hat{\otimes}_A A_{i_1} \hat{\otimes}_A A_{i_2} \to \cdots \to M \hat{\otimes}_A A_1 \hat{\otimes}_A A_2 \hat{\otimes}_A \cdots \hat{\otimes}_A A_n \to 0 . \tag{3.3}$$

Proof. It follows from the same proof as [BBK17, Lemma 5.34 and Remark 5.35]. We give a sketch for the convenience of the readers. When $M = A$, we can prove (3.3) exactly as in the affinoid setting, namely it suffices to treat the case where the covering is given by $\{A\{f\}, A\{f^{-1}\}\}$ for some $f \in A$. Then the acyclicity follows from a direct computation. See [BGR84, Chapter 8] for details. For a general M, taking derived tensor product with (3.3) for $M = A$ and apply the transversality condition, we get (3.3) for M. □

Corollary 3.18. *Let $X = \operatorname{Sp} A$ be a Liu k-analytic space. Let $\mathcal{M}$ be a quasi-coherent sheaf on X. Let $M = H^0(X, \mathcal{M})$. Then for any Liu domain $\operatorname{Sp} B$ in X, we have*

$$H^0(\operatorname{Sp} B, \mathcal{M}) = M \hat{\otimes}_A B .$$

Corollary 3.19. *Let $X = \operatorname{Sp} A$ be a Liu k-analytic space. Let $\mathcal{M}$ be a quasi-coherent sheaf on X. Then*

$$H^i(X, \mathcal{M}) = 0 , \quad i > 0 .$$

Proof. This follows from [Stacks, Tag 01EW]* and Theorem 3.17. □

Definition 3.10. Let X be a k-analytic space. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules (resp. $\mathcal{O}_X$-algebras). A *Banach structure* on $\mathcal{F}$ is the following data: given any Liu domain $\operatorname{Sp} A$ in X, $\mathcal{F}(\operatorname{Sp} A)$ is topologized so that it forms a Banach A-module (resp. Banach A-algebra). We assume that the following condition holds: if $\operatorname{Sp} A$, $\operatorname{Sp} B$ are Liu domains in X such that $\operatorname{Sp} A$ is an analytic domain of $\operatorname{Sp} B$, then the natural morphism of A-modules (resp. A-algebras) $\mathcal{F}(\operatorname{Sp} B)\hat{\otimes}_B A \to \mathcal{F}(\operatorname{Sp} A)$ is bounded.

An $\mathcal{O}_X$-module(resp. $\mathcal{O}_X$-algebra) with a given Banach structure is called a *sheaf of Banach modules* (resp. *sheaf of Banach algebras*) on X.

A morphism $\mathcal{F} \to \mathcal{G}$ of sheaves of Banach modules (resp. sheaves of Banach algebras) on X is a morphism of the underlying sheaves of modules (resp. sheaves of algebras) such that for each Liu domain $\operatorname{Sp} B$ in X, $\mathcal{F}(\operatorname{Sp} B) \to \mathcal{G}(\operatorname{Sp} B)$ is bounded.

The category of sheaves of Banach modules on X is denoted by $\mathcal{B}\mathrm{an}\mathcal{M}\mathrm{od}_X$.

Proposition 3.20. *Let $X = \operatorname{Sp} A$ be a Liu k-analytic space. Let $\mathcal{M}$ be a quasi-coherent sheaf on X. Let $M = H^0(X, \mathcal{M})$. Let $\mathcal{F}$ be a sheaf of Banach $\mathcal{O}_X$-modules. Then*

$$\operatorname{Hom}_{\mathcal{B}\mathrm{an}\mathcal{M}\mathrm{od}_X}(\mathcal{M}, \mathcal{F}) = \operatorname{Hom}_{\mathcal{B}\mathrm{an}\mathcal{M}\mathrm{od}_A}(M, H^0(X, \mathcal{F})) .$$

Proof. Given a morphism $f : \mathcal{M} \to \mathcal{F}$ in $\mathcal{B}\mathrm{an}\mathcal{M}\mathrm{od}_X$, by taking global sections, we get $H^0(f) : M \to H^0(X, \mathcal{F})$. Conversely, given a bounded homomorphism $F : M \to H^0(X, \mathcal{F})$, we construct the morphism of sheaves $f : \mathcal{M} \to \mathcal{F}$ as follows: for any affinoid domain $\operatorname{Sp} B$ in X, define $f(\operatorname{Sp} B) : M\hat{\otimes}_A B \to H^0(\operatorname{Sp} B, \mathcal{F})$ as the natural homomorphism of Banach B-modules induced by the homomorphism of Banach A-modules:

$$M \xrightarrow{F} H^0(X, \mathcal{F}) \to H^0(\operatorname{Sp} B, \mathcal{F}) .$$

By the obvious functoriality, this is a morphism of Banach $\mathcal{O}_X$-modules. It is easy to verify that these maps are inverse to each other. □

Theorem 3.21. *Let $f : \operatorname{Sp} B \to \operatorname{Sp} A$ be a morphism in $\mathcal{L}\mathrm{iu}_k$. Let $\mathcal{M}$ be a quasi-coherent sheaf on $\operatorname{Sp} B$. Then $f_*\mathcal{M}$ is a quasi-coherent sheaf on $\operatorname{Sp} A$ associated to the transversal A-module $H^0(\operatorname{Sp} B, \mathcal{M})$.*

Proof. Let $F : A \to B$ be the corresponding homomorphism of Liu k-algebras. Let $M = H^0(\operatorname{Sp} B, \mathcal{M})$. We claim that M is transversal as Banach A-module.

This is proved in [BBK17, Lemma 4.48], we reproduce the argument: let $\operatorname{Sp} D \to \operatorname{Sp} A$ be a Liu domain. We need to show that

$$M\hat{\otimes}^{\mathbb{L}}_A D = M\hat{\otimes}_A D .$$

Observe that

$$M\hat{\otimes}^{\mathbb{L}}_A D = M\hat{\otimes}^{\mathbb{L}}_B (B\hat{\otimes}^{\mathbb{L}}_A D) = M\hat{\otimes}^{\mathbb{L}}_B (B\hat{\otimes}_A D) = M\hat{\otimes}_B (B\hat{\otimes}_A D) = M\hat{\otimes}_A D ,$$

where for the second equality, we have applied Theorem 3.16; for the third we used Lemma 3.9 and the transversality of M. This concludes the claim.

In order to prove the theorem, it suffices to show $\widetilde{M^A} = f_*\mathcal{M}$. Here M^A is M regarded as a Banach A-module. To prove this, it suffices to take an affinoid domain $\operatorname{Sp} C$ in $\operatorname{Sp} A$ and show that

$$M\hat{\otimes}_A C = \mathcal{M}(f^{-1} \operatorname{Sp} C) . \tag{3.4}$$

*This result is only stated for a ringed space, but it is easy to check that the proof works in the current situation as well.

By Lemma 3.9, $f^{-1}\operatorname{Sp} C$ is a Liu domain in $\operatorname{Sp} B$ and $f^{-1}\operatorname{Sp} C = \operatorname{Sp}(B\hat{\otimes}_A C)$. Hence (3.4) follows from Corollary 3.18. □

Lemma 3.22. *Let A be a Liu k-algebra. Consider an admissible exact sequence*

$$0 \to F \to G \to H$$

in $\mathcal{B}\mathrm{an}\mathcal{M}\mathrm{od}_A$. Assume that G, H are both transversal, then so is F.

This is clear by definition.

Corollary 3.23. *Let $f : Y \to X$ be a quasi-compact and quasi-separated morphism of k-analytic spaces. Assume that $X = \operatorname{Sp} A$ is Liu. Let $\mathcal{F}$ be a Banach sheaf of $\mathcal{O}_Y$-modules such that for each affinoid domain $\operatorname{Sp} C$ in Y, $\mathcal{F}|_{\operatorname{Sp} C}$ is quasi-coherent. Then $f_*\mathcal{F}$ is quasi-coherent on X.*

Proof. Let $\{U_i = \operatorname{Sp} B_i\}$ be a finite affinoid covering of Y. For each i, j, let $U_{ij} = U_i \cap U_j$, take a finite affinoid covering $\{U_{ijk}\}$ of U_{ij}. Let f_i (resp. f_{ijk}) be the restriction of f to U_i (resp. U_{ijk}). Then $f_{i*}\mathcal{F}$ (resp. $f_{ijk*}\mathcal{F}$) is the quasi-coherent sheaf associated to $\mathcal{F}(U_i)$ (resp. $\mathcal{F}(U_{ijk})$) by Theorem 3.21. In particular, $\mathcal{F}(U_i)$ (resp. $\mathcal{F}(U_{ijk})$) is a transversal Banach A-module.

There is an admissible exact sequence

$$0 \to \mathcal{F}(Y) \to \prod_i \mathcal{F}(U_i) \to \prod_{i,j,k} \mathcal{F}(U_{ijk}) \,.$$

Thus $\mathcal{F}(Y)$ is a transversal Banach A-modules by Lemma 3.22. In particular, for any affinoid domain $\operatorname{Sp} B$ in X, we have an admissible exact sequence

$$0 \to \mathcal{F}(Y)\hat{\otimes}_A B \to \prod_i \mathcal{F}(U_i)\hat{\otimes}_A B \to \prod_{i,j,k} \mathcal{F}(U_{ijk})\hat{\otimes}_A B \,.$$

By our assumption and Corollary 3.18, this sequence can be rewritten as

$$0 \to \mathcal{F}(Y)\hat{\otimes}_A B \to \prod_i \mathcal{F}(U_i \cap f^{-1}(\operatorname{Sp} B)) \to \prod_{i,j,k} \mathcal{F}(U_{ijk} \cap f^{-1}(\operatorname{Sp} B)) \,.$$

It is now clear that $\widetilde{\mathcal{F}(Y)}^A = f_*\mathcal{F}$ and $f_*\mathcal{F}$ is quasi-coherent. □

Theorem 3.24. *Let $f : Y = \operatorname{Sp} B \to X = \operatorname{Sp} A$ be a morphism in $\mathcal{L}\mathrm{iu}_k$. Let $\mathcal{F}$ be a quasi-coherent sheaf on X. Let $F = H^0(X, \mathcal{F})$. Assume that $\mathcal{F}$ is transversal to f:*

$$F\hat{\otimes}^{\mathbb{L}}_A C = F\hat{\otimes}_A C$$

for all Liu domains $\operatorname{Sp} C$ in $\operatorname{Sp} B$. Then the left adjoint f^* of $f_* : \mathcal{B}\mathrm{an}\mathcal{M}\mathrm{od}_Y \to \mathcal{B}\mathrm{an}\mathcal{M}\mathrm{od}_X$ at $\mathcal{F}$ exists and $f^*\mathcal{F}$ is the quasi-coherent sheaf associated to $F\hat{\otimes}_A B$.

Proof. We claim that $F\hat{\otimes}_A B$ is a transversal Banach B-module.

This is proved in [BBK17, Lemma 4.48], we reproduce their proof: let $\operatorname{Sp} C \to \operatorname{Sp} B$ be a Liu domain, we need to show

$$(F\hat{\otimes}_A B)\hat{\otimes}^{\mathbb{L}}_B C = (F\hat{\otimes}_A B)\hat{\otimes}_B C \,.$$

In fact,

$$(F\hat{\otimes}_A B)\hat{\otimes}^{\mathbb{L}}_B C = (F\hat{\otimes}^{\mathbb{L}}_A B)\hat{\otimes}^{\mathbb{L}}_B C = F\hat{\otimes}^{\mathbb{L}}_A C = F\hat{\otimes}_A C = (F\hat{\otimes}_A B)\hat{\otimes}_B C \,,$$

which concludes the claim.

By Proposition 3.20, for any sheaf of Banach $\mathcal{O}_Y$-modules $\mathcal{G}$,

$$\operatorname{Hom}_{\mathcal{B}\mathrm{an}\mathcal{M}\mathrm{od}_Y}(\widetilde{F\hat{\otimes}_A B}, \mathcal{G}) = \operatorname{Hom}_{\mathcal{B}\mathrm{an}\mathcal{M}\mathrm{od}_B}(F\hat{\otimes}_A B, H^0(Y, \mathcal{G})) = \operatorname{Hom}_{\mathcal{B}\mathrm{an}\mathcal{M}\mathrm{od}_A}(F, H^0(Y, \mathcal{G})) \,.$$

On the other hand, by Proposition 3.20, we have

$$\operatorname{Hom}_{\mathcal{B}\mathrm{an}\mathcal{M}\mathrm{od}_X}(\mathcal{F}, f_*\mathcal{G}) = \operatorname{Hom}_{\mathcal{B}\mathrm{an}\mathcal{M}\mathrm{od}_A}(F, H^0(Y, \mathcal{G})) \,.$$

We conclude. □

4. Liu morphisms and quasi-coherent sheaves of Liu algebras

Let k be a complete non-Archimedean valued field.

4.1. Liu morphisms.

Definition 4.1. Let $f : X \to Y$ be a morphism in $\mathcal{A}\mathrm{n}_k$. We say f is *Liu* if for any Liu domain Z of Y, $f^{-1}Z$ is a Liu domain.

For any k-analytic space Y, let $\mathcal{L}\mathrm{iu}_{\to Y,k}$ denote the category of Liu morphisms $X \to Y$. A morphism between two Liu morphisms $X_1 \to Y$ and $X_2 \to Y$ is a morphism of in the over-category $\mathcal{A}\mathrm{n}_{k/Y}$.

The following two propositions are obvious.

Proposition 4.1. *Let $f : X \to Y$, $g : Y \to Z$ be morphisms in $\mathcal{A}\mathrm{n}_k$. Assume that f, g are both Liu, then so is $g \circ f$.*

Proposition 4.2. *Let $f : X \to Y$ be a Liu morphism in $\mathcal{A}\mathrm{n}_k$. Then f is separated and quasi-compact.*

Lemma 4.3. *Let $f : X \to Y$ be a morphism in $\mathcal{L}\mathrm{iu}_k$. Let $\mathcal{F}$ be a coherent sheaf on X. Then $R^i f_* \mathcal{F} = 0$ for all $i > 0$.*

Proof. The problem is local, so it suffices to show that $H^i(f^{-1}(\mathrm{Sp}\, A), \mathcal{F}) = 0$ for any affinoid domain $\mathrm{Sp}\, A$ of Y. This follows from the fact that $f^{-1}(\mathrm{Sp}\, A)$ is Liu (Proposition 3.4). □

Theorem 4.4. *Let $f : X \to Y$ be a morphism in $\mathcal{A}\mathrm{n}_k$. Assume that Y is Liu. Then the following are equivalent:*

(1) f is Liu.
(2) f is quasi-compact and separated, for any analytic field extension k'/k, and coherent sheaf $\mathcal{F}$ on $X_{k'}$,

$$R^i f_{k'*} \mathcal{F} = 0\,, \quad i > 0\,.$$

(3) X is Liu.

Proof. (1) $\implies$ (2): We may assume that $k' = k$ and it suffices to prove that for any affinoid domain $\mathrm{Sp}\, A$ in Y, $H^i(f^{-1}(\mathrm{Sp}\, A), \mathcal{F}) = 0$ for all $i > 0$, which is trivial as $f^{-1}(\mathrm{Sp}\, A)$ is Liu.

(2) $\implies$ (3): This follows from Leray's spectral sequence.

(3) $\implies$ (1): This follows from Proposition 3.4. □

Example 4.1. *Recall [Day21, Définition 3.18]: A morphism $f : X \to Y$ in $\mathcal{A}\mathrm{n}_k$ is said to be* almost affinoid *(*presque affinoïde *in French) if there is a G-covering of Y by affinoid domains $\{U_i\}$ such that $f^{-1}U_i$ is affinoid for each i.*

An almost affinoid morphism is not necessarily Liu even if the target is affinoid. See [SW20, Example 9.1.2] for a counterexample. I would like to thank Marco Maculan for pointing this out to me.

4.2. Quasi-coherent sheaves of Liu algebras.

Definition 4.2. Let X be a k-analytic space. A sheaf of Banach algebras $\mathcal{F}$ on X is a *quasi-coherent sheaf of Liu k-algebras* if for each Liu domain $\mathrm{Sp}\, A$ in X, $H^0(\mathrm{Sp}\, A, \mathcal{F})$ is a Liu k-algebra and $\mathcal{F}|_{\mathrm{Sp}\, A}$ is a quasi-coherent sheaf (in the sense of Definition 3.9). A morphism of quasi-coherent sheaves of Liu k-algebras on X is a homomorphism of the underlying sheaves of $\mathcal{O}_X$-algebras. We denote the category of quasi-coherent sheaves of Liu k-algebras on X by $\mathcal{L}\mathrm{iu}\mathcal{A}\mathrm{lg}^{\mathrm{QCoh}}_{X,k}$.

Remark 4.1. By Corollary 3.7, a sheaf of Liu k-algebras admits a natural Banach structure. Moreover, a morphism of quasi-coherent sheaves of Liu k-algebras on X is automatically a morphism in $\mathcal{B}\mathrm{an}\mathcal{M}\mathrm{od}_X$. Hence $\mathcal{L}\mathrm{iu}\mathcal{A}\mathrm{lg}^{\mathrm{QCoh}}_{X,k}$ is a full subcategory of $\mathcal{B}\mathrm{an}\mathcal{M}\mathrm{od}_X$.

Remark 4.2. We do not define a quasi-coherent sheaf on a k-analytic space. In fact, according to the philosophy of [BBK17], in the global setting, the correct notion to consider is the derived category of quasi-coherent sheaves.

Proposition 4.5. *Let X be a* separated *k-analytic space. Let $\mathcal{A}$ be a quasi-coherent sheaf of Liu k-algebras on X. Consider the presheaf F on $\mathcal{A}\mathrm{n}_k$:*

$$T \mapsto \{\, (f, \varphi) : f \in \mathrm{Hom}_{\mathcal{A}\mathrm{n}_k}(T, X), \varphi \in \mathrm{Hom}_{\mathcal{O}_T}(f^*\mathcal{A}, \mathcal{O}_T) \,\}\,.$$

Then F is representable.

Proof. Assume first that X is paracompact. It suffices to verify that the conditions of Theorem 2.1 are satisfied.

(1) The sheaf condition follows from [Ber93, Proposition 1.3.2].

(2) Take a locally finite affinoid covering $\{U_i\}$ of X. Observe that each U_i is closed as X is separated. Take F_i to be the subfunctor of F consisting of pairs $(f : T \to S, \varphi)$ such that $f(T) \subseteq U_i$. Then F_i is represented by $\operatorname{Sp} \mathcal{A}(U_i)$. Thus 2(a) is satisfied. The conditions 2(b) and 2(c) follows from the choice of U_i.

In general, take a paracompact open covering $\{V_i\}$ of X as in the final step of [Ber93, Proof of Proposition 1.4.1]. Repeat the same construction as in the previous step, with $\{V_i\}$ in place of $\{U_i\}$, we get subfunctors F_i of F. Again, it suffices to verify the conditions of 2(a), 2(b), 2(c) of Theorem 2.1. The conditions 2(b), 2(c) follows from the choice of $\{V_i\}$, while the condition 2(a) follows from the special we just treated. □

Remark 4.3. Of course, in Proposition 4.5, one can weaken the separateness assumption to Hausdorff condition. It is not clear to the author if one can remove this condition.

Definition 4.3. Let X be a *separated* k-analytic space. Let $\mathcal{A}$ be a quasi-coherent sheaf of Liu k-algebras on X. We define the *relative spectrum* $\underline{\operatorname{Sp}}_X \mathcal{A}$ as the k-analytic space representing the presheaf F in Proposition 4.5. Note that there is a natural morphism $\pi : \underline{\operatorname{Sp}}_X \mathcal{A} \to X$. We sometimes call π the relative spectrum as well.

Proposition 4.6. *Let X be a separated k-analytic space. Let $\mathcal{A}$ be a quasi-coherent sheaf of Liu k-algebras on X. Let $\pi : \underline{\operatorname{Sp}}_X \mathcal{A} \to X$ be the relative spectrum, then*

1. *For each Liu domain $\operatorname{Sp} A$ in X, the restriction of π to $\pi^{-1}(\operatorname{Sp} A) \to \operatorname{Sp} A$ is the same as $\operatorname{Sp} H^0(\operatorname{Sp} A, \mathcal{A}) \to \operatorname{Sp} A$.*
2. *For any morphism of separated k-analytic spaces $g : X' \to X$, $g^*\mathcal{A}$ is a quasi-coherent sheaf of Liu k-algebras and the natural morphism*
$$X' \times_X \underline{\operatorname{Sp}}_X \mathcal{A} \to \underline{\operatorname{Sp}}_{X'} g^*\mathcal{A}$$
is an isomorphism over X'.
3. *The universal map*
$$\mathcal{A} \to \pi_* \mathcal{O}_{\underline{\operatorname{Sp}}_X \mathcal{A}}$$
is an isomorphism of sheaves of Banach algebras on X.

We omit the straightforward proof. See [Stacks, Tag 01LQ] for example.

Corollary 4.7. *Let X be a separated k-analytic space. Then the functor*
$$\underline{\operatorname{Sp}}_X : \mathcal{L}\mathrm{iu}\mathcal{A}\mathrm{lg}^{\mathrm{QCoh}}_{X,k} \to \mathcal{L}\mathrm{iu}_{\to X,k}$$
is an anti-equivalence of categories. The quasi-inverse is given by $f \mapsto f_*$.

5. Quasi-Liu morphisms

Let k be a complete non-Archimedean valued field.

Definition 5.1. A k-analytic space X is called *quasi-Liu* if the following conditions hold:

1. X is quasi-compact.
2. $H^0(X, \mathcal{O}_X)$ is a Liu k-algebra.
3. There is a Liu k-analytic space $\operatorname{Sp} B$ and a morphism $i : X \to \operatorname{Sp} B$, which realizes X as an analytic domain in $\operatorname{Sp} B$.

Proposition 5.1. *Let X be a quasi-Liu k-analytic space. Then the natural morphism $X \to \operatorname{Sp} H^0(X, \mathcal{O}_X)$ is an analytic domain embedding.*

Proof. Let $Y = \operatorname{Sp} B$ be a Liu k-analytic space such that there is a morphism $i : X \to Y$, which is an analytic domain embedding. Now we have a natural homomorphism $B \to A$ given by the restriction map $B = H^0(Y, \mathcal{O}_Y) \to A = H^0(X, \mathcal{O}_X)$. In particular, we get a factorization $X \to \operatorname{Sp} A \to Y$ of i by Theorem 3.8. Now it remains to show that $X \to \operatorname{Sp} A$ is an analytic domain. Take $x \in X$. We can find rational domains $V_1, \dots, V_m$ of Y contained in X such that $x \in \cap_i V_i$ and $\cup_i V_i$ is a neighborhood of x in Y. Let U_i be the rational domain of $\operatorname{Sp} A$ induced by V_i. We claim that $U_i \subseteq X$. Assuming this claim, then we find that $x \in \cup_i U_i$ and $\cup_i U_i \subseteq X$ is a neighborhood of x in $\operatorname{Sp} A$. We conclude that $X \to \operatorname{Sp} A$ is indeed an analytic domain.

To prove the claim, we will fix some i and omit the indices from V_i, U_i. We write $V = \operatorname{Sp} B\{r^{-1}f/g\}$, where $f = (f_1, \dots, f_n)$ is a tuple of elements in B, $r = (r_1, \dots, r_n)$ is a tuple of positive real numbers and g is an element in B such that f_j, g do not have a common zero. Then $U = \operatorname{Sp} A\{r^{-1}f/g\}$. Let X' denote the analytic domain of X consisting of points where $|f_j| \leq r_j|g|$ for all $j = 1, \dots, n$. As $V \subseteq X$, we could identify X' with the analytic domain in Y defined by the same inequalities. In particular, X' is a Liu space. Take a finite affinoid covering $\operatorname{Sp} A_i$ of X, we know that A is the equalizer of $\prod_i A_i \rightrightarrows \prod_{i,j} A_{ij}$, where $\operatorname{Sp} A_{ij} = \operatorname{Sp} A_i \cap \operatorname{Sp} A_j$. By Theorem 3.16, $A\{r^{-1}f/g\}$ is the equalizer of $\prod_i A_i\{r^{-1}f/g\} \rightrightarrows \prod_{i,j} A_{ij}\{r^{-1}f/g\}$. As $\operatorname{Sp} A_i\{r^{-1}f/g\}$ is an affinoid covering of X', we find an isomorphism $H^0(X', \mathcal{O}_{X'}) \cong A\{r^{-1}f/g\}$. It induces an isomorphism $X' \to U$ by Theorem 3.8, which is the inverse of the composition $U \to V \to X'$. In particular, we find that $U \to X$ is injective. □

Lemma 5.2. *Let $f : X \to Y$ be a morphism in $\mathcal{A}\mathrm{n}_k$. Assume that Y is Liu and X is quasi-Liu. Let $g : Y' \to Y$ be a Liu domain in Y. Then $X' := X \times_Y Y'$ is also quasi-Liu.*

Proof. Let $f' : X' \to Y'$ be the base change of f. It suffices to show that $H^0(X', \mathcal{O}_{X'})$ is a Liu k-algebra. By decomposing $X \to Y$ as in the proof of Proposition 5.1, we have the commutative diagram:

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y' \times_Y \operatorname{Sp} H^0(X, \mathcal{O}_X) & \longrightarrow & \operatorname{Sp} H^0(X, \mathcal{O}_X) \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Y \end{array} .$$

Replacing Y by $\operatorname{Sp} H^0(X, \mathcal{O}_X)$ and Y' by $Y' \times_Y \operatorname{Sp} H^0(X, \mathcal{O}_X)$, we may assume that $H^0(X, \mathcal{O}_X) = H^0(Y, \mathcal{O}_Y)$ and f is the analytic domain embedding $X \to H^0(X, \mathcal{O}_X)$ in Proposition 5.1.

We have the following commutative diagram:

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & \square & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array} .$$

Take a finite affinoid G-covering X_i of X, then we get an admissible exact sequence

$$0 \to H^0(Y, \mathcal{O}_Y) \to \prod_i H^0(X_i, \mathcal{O}_X) \to \prod_{i,j} H^0(X_{ij}, \mathcal{O}_X)\,,$$

where $X_{ij} := X_i \cap X_j$. Taking the derived tensor $\hat{\otimes}^{\mathbb{L}}_{H^0(Y,\mathcal{O}_Y)} H^0(Y', \mathcal{O}_{Y'})$ and applying Theorem 3.16 and (3.1), we get an admissible exact sequence

$$0 \to H^0(Y', \mathcal{O}_{Y'}) \to \prod_i H^0(g'^{-1}(X_i), \mathcal{O}_{X'}) \to \prod_{i,j} H^0(g'^{-1}(X_{ij}), \mathcal{O}_{X'})\,.$$

In particular,

$$H^0(Y', \mathcal{O}_{Y'}) = H^0(X', \mathcal{O}_{X'})$$

and this algebra is a Liu algebra. Also observe that the morphism $f' : X' \to Y'$ satisfies the assumption of Definition 5.1(3) and X' is quasi-Liu. □

Definition 5.2. Let $f : X \to Y$ be a morphism of k-analytic spaces. We say f is *quasi-Liu* if for any Liu domain Z in Y, $f^{-1}Z$ is quasi-Liu.

Proposition 5.3. *Let $f : X \to Y$ be a quasi-Liu morphism in $\mathcal{A}\mathrm{n}_k$. Then f is separated and quasi-compact.*

This is obvious.

Proposition 5.4. *Let $f : X \to Y$ be a morphism of k-analytic spaces. Assume that Y is separated. The following are equivalent:*

(1) f is quasi-Liu.
(2) $f_*\mathcal{O}_X$ is a quasi-coherent sheaf of Liu k-algebras and the natural morphism $X \to \underline{\mathrm{Sp}}_Y f_*\mathcal{O}_X$ is quasi-compact and realizes X as an analytic domain.
(3) $f_*\mathcal{O}_X$ is a quasi-coherent sheaf of Liu k-algebras on Y and X can be realized as an analytic domain in $\underline{\mathrm{Sp}}_Y \mathcal{A}$ through a quasi-compact morphism $X \to \underline{\mathrm{Sp}}_Y \mathcal{A}$ over Y, where $\mathcal{A}$ is a quasi-coherent sheaf of Liu k-algebras on Y.

Proof. It is clear that (2) $\implies$ (3) $\implies$ (1).

(1) $\implies$ (2): Observe that $f_*\mathcal{O}_X$ is quasi-coherent by Corollary 3.23. It is a quasi-coherent sheaf of Liu k-algebras by Lemma 5.2. The last assertion follows from Proposition 5.1. □

Proposition 5.5. *Let $f : X \to Y$, $g : Y \to Z$ be morphisms in $\mathcal{A}\mathrm{n}_k$. If f is quasi-Liu and g is Liu, then $g \circ f$ is quasi-Liu.*

Proof. We need to show that the inverse image of a Liu domain U in Z by $g \circ f$ is quasi-Liu. But $g^{-1}(U)$ is Liu and we find that $f^{-1}(g^{-1}(U))$ is quasi-Liu by definition. □

6. Open problems

Let k be a complete non-Archimedean valued field.

We give a list of unsolved problems related to Liu k-algebras and Liu morphisms.

Question 6.1. *Is there a global version of Zariski's main theorem in non-Archimedean geometry?*

A local version is proved by Ducros in [Duc07, Théorème 3.2] based on Temkin's graded reduction. This theorem roughly says that a quasi-finite morphism of separated k-analytic spaces can be written locally as the composition of an étale morphism, an analytic domain embedding and a finite morphism. This theorem, however, does not tell us much information about the global structure of a quasi-finite morphism, in contrast to the classical Zariski's main theorem ([Stacks, Tag 02LR]).

We would like to know if the following holds:

Conjecture 6.1. *Let $f : X \to S$ be a quasi-finite morphism of quasi-compacted, separated k-analytic spaces. Then we can decompose f into $h \circ i \circ g$, where $g : X \to Y$ is finite, $i : Y \to Z$ is a quasi-compact analytic domain embedding, $h : Z \to S$ is étale.*

We hope to find suitable extra conditions on f, which guarantee that i is a Liu domain embedding as well.

Question 6.2. *Are Liu k-algebras excellent?*

In the case of affinoid algebras, this is proved by Ducros [Duc09]. The author is not sure if Ducros' argument can be generalized to the current setting.

Question 6.3. *Can Liu morphisms be effectively descended with respect to fpqc (or Tate-flat) coverings?*

In a previous version of this paper, the author claimed a proof. But as pointed out by the referee, the proof contains a gap. By [Day21, Théorème A], the essential difficulty is to treat the case of descending along a finite faithfully flat morphism of affinoid spaces.

Appendix A. Results from Ben-Bassat–Kremnizer

We slightly generalize a few results in [BBK17].

Definition A.1. Let $f : A \to B$ be a morphism in $\mathcal{B}\mathrm{an}\mathcal{A}\mathrm{lg}_k$. We say f is a *homotopy epimorphism* if the following equivalent conditions are satisfied

(1) $\mathbb{L}f_* : \mathrm{D}^-(B) \to \mathrm{D}^-(A)$ is fully faithful.
(2) The natural morphism
$$\mathbb{L}f^* \circ \mathbb{L}f_* \to \mathrm{id}_{\mathrm{D}^-(B)}$$
is a natural equivalence.
(3) $B\hat{\otimes}^{\mathbb{L}}_A B = B$.

Definition A.2. Let $f : \operatorname{Sp} A \to \operatorname{Sp} B$ be a morphism in $\mathcal{L}\mathrm{iu}_k$. We say f is a *homotopy monomorphism* if the corresponding morphism $B \to A$ in $\mathcal{L}\mathrm{iu}\mathcal{A}\mathrm{lg}_k$ is a homotopy epimorphism (Definition A.1).

Lemma A.1. *Let $A \to B$ be a morphism in $\mathcal{L}\mathrm{iu}\mathcal{A}\mathrm{lg}_k$. For any $r > 0$, $f \in A$, we have the natural isomorphisms in $\mathrm{D}^-(A)$:*

$$B\hat{\otimes}^{\mathbb{L}}_A A\{r^{-1}f\} \to B\hat{\otimes}_A A\{r^{-1}f\}\,, \quad B\hat{\otimes}^{\mathbb{L}}_A A\{rf^{-1}\} \to B\hat{\otimes}_A A\{rf^{-1}\}\,.$$

Proof. We only treat the former. As in the case of affinoid algebras ([BBK17, Lemma 5.13]), it suffices to prove that the morphism

$$T - f : A\{r^{-1}f\} \to A\{r^{-1}f\}$$

is a strict monomorphism. That this morphism is a monomorphism is well-known (and can be proved exactly as in the affinoid case).

To see $T - f$ is strict, by [Ber12, Proposition 2.1.2], we could assume that k is non-trivially valued. Then the image of $T - f$ is closed by Proposition 3.6. Hence $T - f$ is strict. □

Lemma A.2. *Let $A \to B$ be a morphism in $\mathcal{L}\mathrm{iu}\mathcal{A}\mathrm{lg}_k$. Let $f_1, \dots, f_n, g \in A$ be elements that generate A. Let $r_1, \dots, r_n \in \mathbb{R}_{>0}$, Then we have the natural isomorphism in $\mathrm{D}^-(A)$:*

$$B\hat{\otimes}^{\mathbb{L}}_A A\{r_i^{-1}f_i/g\} \to B\hat{\otimes}_A A\{r_i^{-1}f_i/g\}\,.$$

The proof goes exactly as [BBK17, Lemma 5.14].

Lemma A.3. *Let A be a Liu k-algebra. Let A_1, A_2, B be Liu k-algebras over A. Assume that*

(1) $\operatorname{Sp} A_i \to \operatorname{Sp} A$ $(i = 1, 2)$ are Liu domains.
(2) $\operatorname{Sp} A_1 \cup \operatorname{Sp} A_2$ is also a Liu domain in $\operatorname{Sp} A$ with Liu k-algebra C.
(3) Let A_0 be the Liu k-algebra of the Liu domain $\operatorname{Sp} A_1 \cap \operatorname{Sp} A_2$ (c.f. Corollary 3.5). Then the following natural morphisms are isomorphisms

$$A_i\hat{\otimes}^{\mathbb{L}}_A B \to A_i\hat{\otimes}_A B$$

for $i = 0, 1, 2$.

Then we have a natural isomorphism

$$C\hat{\otimes}^{\mathbb{L}}_A B \to C\hat{\otimes}_A B\,.$$

This is obvious.

Theorem A.4. *Let A be a Liu k-algebra. Let B, C be Liu k-algebras over A such that $\operatorname{Sp} C \to \operatorname{Sp} A$ is a Liu domain. Then we have a natural isomorphism*

$$C\hat{\otimes}^{\mathbb{L}}_A B \to C\hat{\otimes}_A B\,.$$

In particular, $\operatorname{Sp} C \to \operatorname{Sp} A$ is a homotopy monomorphism.

Proof. Having established the three preceding lemmas, the proof is the same as [BBK17, Proof of Theorem 5.16]. □

Theorem A.5. *Let $f : A \to B$ be a morphism in $\mathcal{L}\mathrm{iu}\mathcal{A}\mathrm{lg}_k$. Then f is a homotopy epimorphism iff the corresponding morphism $\operatorname{Sp} B \to \operatorname{Sp} A$ is a Liu domain.*

Proof. Same proof as [BBK17, Theorem 5.31]. □

In terms of [BBK17], we have shown that $\mathcal{L}\mathrm{iu}\mathcal{A}\mathrm{lg}_k$ is a homotopy Zariski transversal subcategory of $\mathcal{B}\mathrm{an}_k$.

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